Mathématiques pour SHS

Master Sciences des données et histoire

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UNIVERSITÉ PARIS 1
PANTHÉON SORBONNE

Notes

5. Graphing and Optimization

Notes

5. Graphing and Optimization

5-1 First Derivative and Graphs

Learning Objectives

- Use the first derivative to determine when functions are increasing or decreasing.
- Use the first derivative test to determine the local extrema of functions.

Notes

5. Graphing and Optimization

5-1 First Derivative and Graphs

Correspondence between behavior of $f'(x)$ at $x = c$ and behavior of graph of $f(x)$ at that $x = c$

- f' is positive at $x = c \iff$ The line tangent to graph of f at $x = c$ exists and it tilts upward.
- f' is negative at $x = c \iff$ The line tangent to graph of f at $x = c$ exists and it tilts downward.
- f' is zero at $x = c \iff$ The line tangent to graph of f at $x = c$ exists and it is horizontal.

Notes

Now talk about behavior on an interval, not just at some particular $x = c$.

DEFINITION

Words: f is increasing on the interval $a < x < b$

Meaning: If $a < x_1 < x_2 < b$ then $f(x_1) < f(x_2)$

Graphical interpretation: If you move from left to right across the interval, the $y-$values go up

DEFINITION

Words: f is decreasing on the interval $a < x < b$

Meaning: If $a < x_1 < x_2 < b$ then $f(x_1) > f(x_2)$

Graphical interpretation: If you move from left to right across the interval, the $y-$values go down

Notes

Correspondence between behavior of f' on an interval and behavior of f on that interval

- f' is positive on interval $a < x < b \iff f$ is increasing on interval $a < x < b$.
- f' is negative on interval $a < x < b \iff f$ is decreasing on interval $a < x < b$.
- f' is zero on whole interval $a < x < b \iff f$ is constant on the whole interval $a < x < b$.

Notes

THEOREM 1: Increasing and Decreasing Functions

On the interval (a, b)		
$f'(x)$	$f(x)$	Graph of f
$+$	increasing	rising
$-$	decreasing	falling

Notes

EXAMPLE 1

Find the intervals where $f(x) = x^2 + 6x + 7$ is rising and falling.
From the previous table, the function will be rising when the derivative is positive.

$$f'(x) = 2x + 6$$

$2x + 6 > 0$ when $2x > -6$, or $x > -3$.

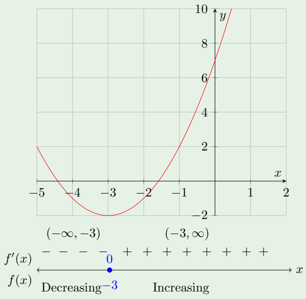
The graph is rising when $x > -3$.

$2x + 6 < 0$ when $x < -3$, so the graph is falling when $x < -3$.

Notes

EXAMPLE 1

A sign chart is helpful:



Notes

Partition Numbers and Critical Values

A partition number for the sign chart is a place where the derivative could change sign. Assuming that f' is continuous wherever it is defined, this can only happen where f itself is not defined, where f' is not defined, or where f' is zero.

DEFINITION **Critical Values**

The values of x in the domain of f where $f'(x) = 0$ or does not exist are called the critical values of f.

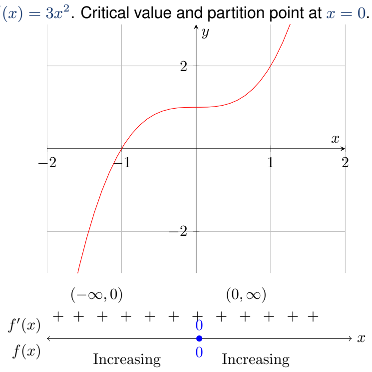
Insight:

All critical values are also partition numbers, but there may be partition numbers that are not critical values (where f itself is not defined).
If f is a polynomial, critical values and partition numbers are both the same, namely the solutions of $f'(x) = 0$.

Notes

EXAMPLE 2

$f(x) = 1 + x^3$, $f'(x) = 3x^2$. Critical value and partition point at $x = 0$.



Notes

Local Extrema

- When the graph of a continuous function changes from rising to falling, a high point or local maximum occurs.
- When the graph of a continuous function changes from falling to rising, a low point or local minimum occurs.

DEFINITION **Local extrema**

Words: f has a local max (min) at $x = c$
Meaning: $f(c)$ exists, $f(c)$ is the highest (lowest) $y-$value nearby. That is for all x near $x = c$, $f(c) \geq (\leq) f(x)$.
We says that the local max (min) occurs at $x = c$, but the value of the local max (min) is the $y-$value $f(c)$.

THEOREM **Existence of Local Extrema**

If f is continuous on the interval (a, b), c is a number in (a, b), and $f(c)$ is a local extremum, then either $f'(c) = 0$ or $f'(c)$ does not exist. That is, c is a critical point.

Notes

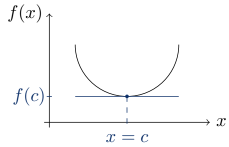
First Derivative Test

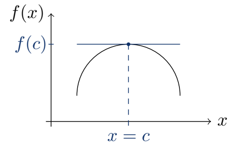
Let c be a critical value of f. That is, $f(c)$ is defined, and either $f'(c) = 0$ or $f'(c)$ is not defined. Construct a sign for $f'(x)$ close to and on either side of c.

On the interval (a, b)		
$f(x)$ left of c	$f(x)$ right of c	$f(c)$
Decreasing	Increasing	local minimum at c
Increasing	Decreasing	local maximum at c
Decreasing	Decreasing	not an extremum
Increasing	Increasing	not an extremum

Notes

$f'(c) = 0$: Horizontal Tangent



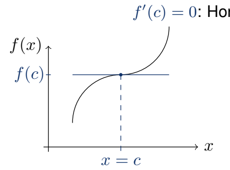


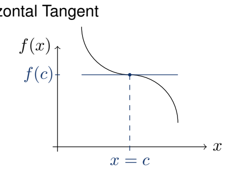
$f'(x)$ $\boxed{- \ - \ -0+ \ + \ +}$ $f'(x)$ $\boxed{+ \ + \ +0- \ - \ -}$

(A) $f(c)$ is a local minimum (B) $f(c)$ is a local maximum

Notes

$f'(c) = 0$: Horizontal Tangent



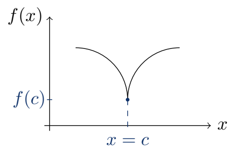


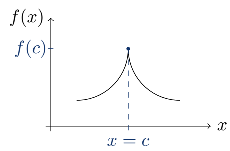
$f'(x)$ $\boxed{+ \ + \ +0+ \ + \ +}$ $f'(x)$ $\boxed{- \ - \ -0- \ - \ -}$

(C) $f(c)$ is neither a local max nor a local min (D) $f(c)$ is neither a local max nor a local min

Notes

$f'(c)$ is not defined but $f(c)$ is defined



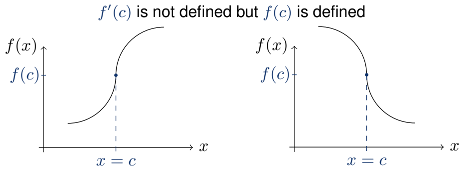


$f'(x)$ $\boxed{- \ - \ -\text{ND}+ \ + \ +}$ $f'(x)$ $\boxed{+ \ + \ +\text{ND}- \ - \ -}$

(E) $f(c)$ is a local minimum (F) $f(c)$ is a local maximum

Notes

$f'(c)$ is not defined but $f(c)$ is defined



$f'(x)$ $\boxed{+\ +\ +\text{ND}+\ +\ +}$ $f'(x)$ $\boxed{-\ -\ -\text{ND}-\ -\ -}$

(G) $f(c)$ is neither a local max nor a local min

(H) $f(c)$ is neither a local max nor a local min

Notes

THEOREM 3 Intercepts and Local Extrema of Polynomial Functions

If

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x_1 + a_0, \quad a_n \neq 0,$$

is an $n^{th}-$degree polynomial, then f has at most n $x-$intercepts and at most $(n-1)$ local extrema.

THEOREM 3 does not guarantee that every $nth-$degree polynomial has exactly $n-1$ local extrema; it says only that there can never be more than $n-1$ local extrema.

Notes

EXERCISES

1. Use a sign graph to determine the intervals where x in increasing or decreasing. Give your answers in interval notation.
   a. $f(x) = 15x^2 - 30x - 60$
   b. $f(x) = 4x^3 - 3x^2$
2. Determine the intervals where $g(x)$ is increasing or decreasing. Identify the critical values of $g(x)$.
   a. $g(x) = \frac{x^3}{3} - x^2 - 15x + 4$
   b. $g(x) = \frac{x^2}{x+4}$
3. Let $f(x) = -x^4 + 50x^2$.
   a. Finds intervals where f is increasing or decreasing. Present the answers three ways: inequality notation and interval notation
   b. Find $x-$coordinates of all local extrema.
   c. Find the $y-$values of the local extrema.
   d. Sketch a graph

Notes

EXERCISES

4. Given that $f(x)$ is continuous on $(-\infty, \infty)$, use the information to sketch a graph of $f(x)$.

$$f(4) = 0, f(1) = 9$$
$$f'(1) = 0, f'(x) > 0, \quad \text{on} \quad (1, \infty)$$
$$f'(x) < 0, \quad \text{on} \quad (-\infty, 1)$$

5. Determine the local extrema for the functions in Exercise 2.

Notes

Learning Objectives

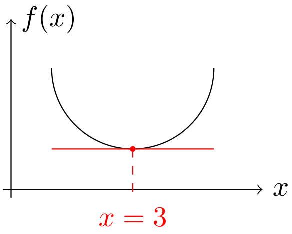
- Use the second derivative to determine the concavity of functions.
- Use the second derivative to determine the inflection points of functions.
- Solve applications involving the point of diminishing returns.

Notes

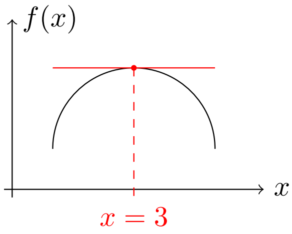
DEFINITION **Concavity at a particular x value**

Words: f is concave up at $x = c$

Meaning: The graph of f has a tangent line at $x = c$ and for $x-$values near $x = c$, the graph of f stays above the tangent line.



f concave up at $x = 3$



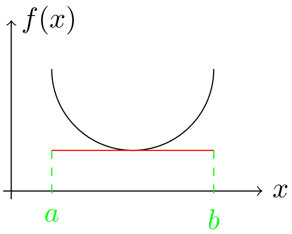
f concave down at $x = 3$

Notes

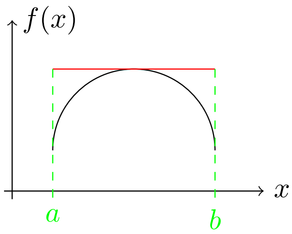
DEFINITION **Concavity on an interval**

Words: f is concave up on an interval $a < x < b$

Meaning: For every $x = c$ where $a < c < b$, f is concave up at $x = c$.



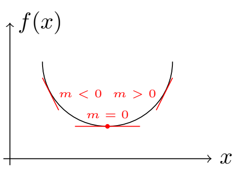
f concave up on (a, b)

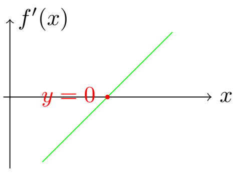


f concave down on (a, b)

Notes

Consider relationship between concavity of f and the behavior of f'





Notes

It seems that

f' increasing on interval $a < x < b \Longleftrightarrow f$ concave up on interval $a < x < b$

Similarly f' decreasing on interval $a < x < b \Longleftrightarrow f$ concave down on interval $a < x < b$

But remember that

A function g being increasing or decreasing on an interval $a < x < b$ is related to the derivative of g being positive or negative on the interval $a < x < b$.

So f' being increasing or decreasing on an interval $a < x < b$ is related to the derivative of f' being positive or negative on the interval $a < x < b$.

This leads us to consider the derivative of f'.

Notes

NOTATION

Introduce the second derivative of f

Symbol: f'' or $f''(x)$ or $\frac{d^2f}{dx^2}$ or y'' or $\frac{d^2y}{dx^2}$

Words: The second derivative of f.

Meaning: The derivative of the derivative of f that is:

$$f''(x) = \frac{d}{dx}\left(\frac{d}{dx}f(x)\right) = \frac{d}{dx}\left(f'(x)\right)$$

EXAMPLE 1

For $f(x) = -x^4 + 50x^2$ and $f(x) = xe^{-x}$ find $f''(x)$

Notes

Relationship between

Sign of $f'' \Longleftrightarrow$ increasing/decreasing behavior of $f' \Longleftrightarrow$ Concavity behavior of f

SUMMARY **Concavity**

For the interval (a, b)

$f''(x)$	$f'(x)$	Graph of $y = f(x)$
$+$	Increasing	Concave up
$-$	Decreasing	Concave down

EXAMPLE 2

Find the intervals where the graph of $f(x) = 2x^5 - 3x^4$ is concave up or concave down.

Notes

DEFINITION **Inflection point**

Inflection point on graph of f is

- a point on the graph
- where the concavity changes.

This means that if $f''(x)$ exists in a neighborhood of an inflection point, then it must change sign at that point.

THEOREM **Inflection point**

If $y = f(x)$ is continuous on (a, b) and has an inflection point at $x = c$, then either $f''(c) = 0$ or $f''(c)$ does not exist

The theorem means that an inflection point can occur only at critical value of f''. However, not every critical value produces an inflection point.

EXAMPLE 3

Find the inflection point(s) of $f(x) = 2x^5 - 3x^4$.

Notes

Analytical Example

Questions: Given a function

1. Find intervals where function is increasing or decreasing.
2. Find $x-$values of local max and min
3. Find $y-$values of the local max and min
4. Find intervals where function is concave up or down
5. Find $x-$values of inflection points
6. Find $y-$values of inflection points

Notes

EXAMPLE 4

Let $f(x) = xe^{-x}$, Answer questions 1-6.

1. To determine increasing or decreasing behavior of f, we should study the sign of f'. So we need $f'(x)$. Here, we have $f'(x) = (1-x)e^{-x}$.
   We need to make a sign chart for $f'(x)$. Start by looking for $x-$values $x = c$ where
   - $f'(c) = 0$
   - $f'(c)$ DNE

   (these are called partion numbers for $f'(x)$)
   Are there any $x = c$ where $f'(c)$ DNE?

$$f'(x) = \underbrace{(1-x)}_{\text{this is a poly, its domain is all x}} \times \underbrace{e^{-x}}_{\text{this is an exp fun, its domain is all x}}$$

   Conculde: The domain of f' is all x. There are no $x = c$ where $f'(c)$ DNE.

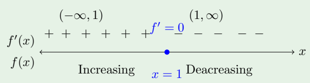
Notes

EXAMPLE 4

Are there any $x-$values where $f'(c) = 0$

$$0 = f'(x)$$
$$0 = \underbrace{(1-x)}_{x=1 \text{ will cause this factor to become zero}} \times \underbrace{e^{-x}}_{e^{anything} > 0 \text{ so no x-values will ever cause } e^{-x} = 0}$$

Conclusion: the only $x-$value that will cause $f'(x) = 0$ is $x = 1$.
Conclude: $x = 1$ is the only partition number for $f'(x)$.
Now make a sign chart for $f'(x)$



$$f'(2) = (1-2)e^{-2} : nes \cdot pos = neg$$
$$f'(0) = (1-0)e^{-0} : pos \cdot pos = pos$$

Notes

EXAMPLE 4

Conclusion of question 1:

- f is increasing on interval $(-\infty, 1)$, because f' is positive there.
- f is decreasing on interval $(1, \infty)$, because f' is negative there.

2. Local max at $x = 1$ because f changes from inc to dec (because f' changes from pos to neg) and because we know that $x = 1$ is a critical value of f.
   That is
   - $x = 1$ is a partition number for f'
   - $f(1)$ exists because domain of f is all real numbers.

   No local min!
3. The $y-$value of the local max. Substitute $x = 1$ into $f(x)$.

$$y = f(1) = 1e^{-1} = \frac{1}{e}$$

4. Strategy:
   - find f''
   - analyze sign of f''
   - use the information about sign of f'' to answer question about concavity of f.

Notes

EXAMPLE 4

We need to analyze the sign of $f''(x) = (x-2)e^{-x}$ (use approach similar to what we did when we analyzed the sign of $f'(x)$) Start by finding partition numbers for $f''(x)$.
Are there any $x-$values $x = c$ such that

- $f''(c)$ DNE, or
- $f''(c) = 0$

Observe

$$f''(x) = \underbrace{(x-2)}_{\text{this is a poly, its domain is all x}} \times \underbrace{e^{-x}}_{\text{this always exists for every x}}$$

So the product always exists for every x. Are there any $x-$values where $f''(c) = 0$?

$$0 = f''(x)$$
$$0 = \underbrace{(x-2)}_{x=2 \text{ will cause this factor to become zero}} \times \underbrace{e^{-x}}_{\text{always pos because } e^{anything} > 0}$$

Conclusion: $f''(x)$ has one partition number $x = c = 2$.

Notes

EXAMPLE 4

$(-\infty, 2)$ $(2, \infty)$

$f''(x)$ $- - - - - -$ $f''=0$ $+ + + + + +$

x

$f'(x)$ Decreasing $x = 2$ Increasing

$$f''(1) = (1-2)e^{-1} = neg \cdot pos = neg$$
$$f''(3) = (3-2)e^{-3} = pos \cdot pos = pos$$

Conclusion:

- f is concave up on the interval $(2, \infty)$
- f is concave down on the interval $(-\infty, 2)$

Notes

EXAMPLE 4

5 Find $x-$values of inflection points.

- We know the concavity changes at $x = 2$.
- We also know that $f(2)$ exists because $f(2) = 2e^{-2}$ this will exist.

So there is a point on graph of f at $x = 2$. Conclude there is an inflection point on graph of f at $x = 2$

6 The $y-$value of the inflection point is:

$$f(2) = 2e^{-2} = \frac{2}{e^2}$$

Notes

Point of Diminishing Returns

If a company decides to increase spending on advertising, they would expect sales to increase.

At first, sales will increase at an increasing rate and then increase at a decreasing rate. The value of x where the rate of change of sales changes from increasing to decreasing is called the point of diminishing returns.

This is also the point where the rate of change has a maximum value. Money spent after this point may increase sales, but at a lower rate. The next example illustrates this concept.

Notes

Maximum Rate of Change Example

Currently, a discount appliance store is selling 200 large-screen television sets monthly. If the store invests $\$x$ thousand in an advertising campaign, the ad company estimates that sales will increase to

$$N(x) = 3x^3 - 0.25x^4 + 200, \quad 0 \le x \le 9$$

- When is rate of change of sales increasing and when is it decreasing?
- What is the point of diminishing returns and the maximum rate of change of sales?

Notes

Maximum Rate of Change Example

The rate of change of sales with respect to advertising expenditures is

$$N'(x) = 9x^2 - x^3 = x^2(9 - x)$$

To determine when $N'(x)$ is increasing and decreasing, we find $N''(x)$, the derivative of $N'(x)$:

$$N''(x) = 18x - 3x^2 = 3x(6 - x)$$

Notes


```
N <- function(x){3*x^3-0.25*x^4+200}
Np <- function(x){9*x^2-x^3}; tgt <- function(x) {108*x-124}
curve(N,0,9,col="blue", ylim=c(0,750)); curve(Np,0,9,add=T)
segments(5,Np(6),7,Np(6)); segments(6,0,6,N(6), lty=2)
segments(5, tgt(5), 7, tgt(7))
points(6,Np(6), pch=16);points(6,N(6), pch=16)
legend("topleft", legend=c("N(x)","N'(x)"),
       col=c("blue","black"), lty=1)
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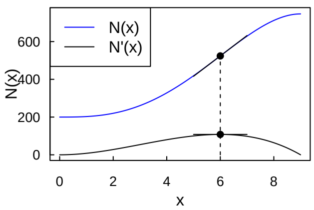


Notes

EXERCISES

1. Find the interval where the graph of f is concave up and concave down. Identify all infection points of $f(x)$.
   a. $f(x) = x^3 - 3x^2 + 2x - 1$
   b. $f(x) = e^{-3x^2}$
   c. $f(x) = \frac{x}{2x-1}$

2. A company estimates that it will sell $N(x)$ units of a product after spending $\$x$ thousand on advertising, as given by

$$N(x) = -0.25x^4 + 13x^3 - 180x^2 + 10{,}000, \quad 15 \le x \le 24$$

- When is rate of change of sales increasing and when is it decreasing?
- What is the point of diminishing returns and the maximum rate of change of sales?
- Graph N and N' on the same coordinate system

Notes

Notes

Learning Objectives

- Use the graphing strategy to sketch the graphs of functions.

Notes

PROCEDURE Graphing Strategy

Step 1 Analyze $f(x)$
1. Find the domain of f.
2. Find the intercepts.
3. Find asymptotes

Step 2 Analyze $f'(x)$
1. Find the partition numbers and critical values of $f'(x)$.
2. Construct a sign chart for $f'(x)$.
3. Determine the intervals where f is increasing and decreasing.
4. Find local maxima and minima.

Step 3 Analyze $f''(x)$
1. Find the partition numbers for $f''(x)$.
2. Construct a sign chart for $f''(x)$.
3. Determine the intervals where f is concave up or down.
4. Find inflection points.

Step 4 Sketch the graph of f
1. Draw asymptotes, local max/min, and inflection points.
2. Plot additional points as needed and complete the sketch.

Notes

EXAMPLE 1

Apply the graphing strategy to sketch the graph of $f(x) = x^3 - 3x^2$.

Step 1 Analyze $f(x)$

1. Domain: the domain of f is all $x-$values (poly).
2. y intercept: if $x = 0$, then $f(0) = 0^3 - 3(0^2) = 0$ is the $y-$intercept
   x intercept: if $y = 0$, then $x^3 - 3x^2 = x^2(x - 3) = 0$ so that $x = 0$ and $x = 3$ are the $x-$intercepts.
3. There are no vertical or horizontal asymptotes since f is a polynomial.

Notes

EXAMPLE 1

Step 2 Analyze $f'(x)$. $f'(x) = 3x^2 - 6x = 3x(x - 2)$

1. Critical values of $f(x) : x = 0$ and $x = 2$.
   Partition numbers for $f'(x) : x = 0$ and $x = 2$.
2. Sign chart for $f'(x)$:

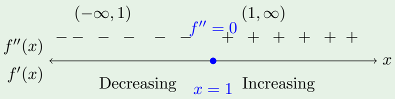
	$(-\infty, 0)$		$(0, 2)$		$(2, \infty)$	
$f'(x)$	$+ + + +$	0	$- - -$	0	$+ + + + +$	x
$f(x)$	Inc	$x = 0$	Dec	$x = 2$	Inc	

3. f increases on $(-\infty, 0)$ and $(2, \infty)$ and decreases on $(0, 2)$.
4. f has a local max at $x = 0$, $y = 0$. f has a local min at $x = 2$, $y = -4$

EXAMPLE 1

Step 3 Analyze $f''(x)$. $f''(x) = 6x - 6 = 6(x-1)$

1. Partition numbers for $f'(x) : x = 1$.
2. Sign chart for $f''(x)$:

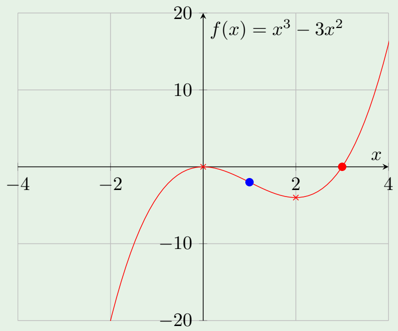


3. f is $\bigcap$ on $(-\infty, 1)$; f is $\bigcup$ on $(1, \infty)$.
4. f has an inflection point at $x = 1, y = -2$

Notes

EXAMPLE 1

Step 4 Sketch the graph of f



Notes

EXAMPLE 2

If x items are produced in one day, the cost per day is

$$C(x) = x^2 + 2x + 2000$$

and the average cost per unit is $C(x)/x$.
Use the graphing strategy to analyze the average cost function.

Step 1 Analyze $\bar{C}(x) = \frac{C(x)}{x} = \frac{x^2+2x+2000}{x}$

1. Domain: Since negative values of x do not make sense and $\bar{C}(0)$ is not defined, the domain is the set of positive real numbers.
2. y intercept: None
   x intercept: None
3. H.A.: None
   V.A.: The line $x = 0$ is a vertical asymptote ($C(0) \neq 0$).

Notes

EXAMPLE 2

Oblique Asymptotes: If a graph approaches a line that is neither horizontal nor vertical as x approaches ∞ or $-\infty$, that line is called an oblique asymptote

$$\bar{C}(x) = \frac{C(x)}{x} = \frac{x^2 + 2x + 2000}{x} = x + 2 + \frac{2000}{x}$$

If x is a large positive number, then $2000/x$ is very small and the graph of $\bar{C}(x)$ approaches the line $y = x + 2$.
This is the oblique asymptote.

Notes

EXAMPLE 2

Step 2 Analyze $\bar{C}'(x)$.

$$\bar{C}'(x) = \frac{(2x+2)x - (x^2+2x+2000)(1)}{x^2} = \frac{x^2-2000}{x^2}$$

1. Critical values of $\bar{C}(x): x = \sqrt{2000} \approx 44.72$.
   Partition numbers for $\bar{C}'(x): x = \sqrt{2000}$ and $x = 0$.
2. Sign chart for $\bar{C}(x)$:

	$(0, 44.72)$		$(44.72, \infty)$	
$\bar{C}'(x)$	$- - - - - -$	0	$+ + + + + +$	x
$\bar{C}(x)$	Decreasing	$x = 44.72$	Increasing	

3. If we test values to the left and right of the critical point, we find that $\bar{C}$ is decreasing on $(0, \sqrt{2000})$, and increasing on $(\sqrt{2000}, \infty)$
4. $\bar{C}$ has a local min at $x = \sqrt{2000}$, $y = 91.44$

Notes

EXAMPLE 2

Step 3 Analyze $\bar{C}''(x)$.

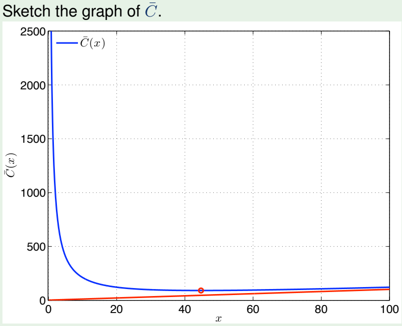
$$\bar{C}''(x) = \frac{2x(x^2) - (x^2-2000)(2x)}{x^4} = \frac{4000}{x^3}$$

Since this is positive for all positive x, the graph of the average cost function is concave up on $(0, \infty)$

Notes

EXAMPLE 2

Step 4 Sketch the graph of $\bar{C}$.



Notes

EXERCISES

1. Summarize the pertinent information obtained by applying the graphing strategy and sketch the graph of $y = f(x)$.
   a. $f(x) = \frac{x^2}{x+1}$
   b. $f(x) = \frac{2x^2-3x}{x-2}$
2. Nicole owns a company that makes luxurious velvet robes. Her total cost to make x robes can be modeled by the function

$$C(x) = 1500 + 3x^2, \quad x > 0.$$

   a. Find the average cost function.
   b. How many robes must be produced for the average cost to be minimized?
   c. What is the minimum average cost?

Notes

Notes

Learning Objectives

- Find the absolute maxima and absolute minima of functions.
- Use the second derivative test for local extrema.

Notes

DEFINITION: **Absolute Maxima and Minima**

- $f(c)$ is an absolute maximum of f if $f(c) > f(x)$ for all x in the domain of f.
- $f(c)$ is an absolute minium of f if $f(c) < f(x)$ for all x in the domain of f.

THEOREM 1

If a function f is continuous on closed interval $[a, b]$, then f is guaranteed to have an absolute max and an absolute min on that interval.

THEOREM 2

The only place where an abs max or min can ever occur (if they occur at all) is at the $x-$values that are

- critical values
- endpoints of the domain

Notes

Suppose that the domain of a function f is a closed interval $[a, b]$.
and suppose that it is known that f is continuous on $[a, b]$.

Theorem 1 guarantees that there will be both an absolute maximum and an absolute minimum on the interval $[a, b]$.

and Theorem 2 tells us where (at what $x-$values) the absolute max and min have to be found.

- at x values that are critical values of f
- at x values that are endpoints ($x = a, x = b$).

This give us the idea for a strategy:

Notes

PROCEDURE **Finding Absolute Extrema on a Closed Interval**

Used for finding the absolute extrema for a function f that is continuous on a closed interval $[a, b]$.

Step 1 Identify the closed interval $[a, b]$.

Step 2 Confirm that f is indeed continuous on the interval $[a, b]$.

Step 3 Find the critical values of f.

Step 4 List all important $x-$values in order in a table.

Step 5 Find the correspond $y-$values.

Step 6 Identify the largest $y-$value as the abs max and the smallest $y-$value as the absolute min. State your conclusion clearly

EXAMPLE 1

Find the absolute extrema of $f(x) = x^4 - 6x^2 + 5$ on the interval $[-3, 2]$.

Step 1 The interval $[-3, 2]$ is a closed interval.

Step 2 The function f is continuous on $[-3, 2]$ because f is a polynomial

Step 3 Critical values of f:
Start by finding $f'(x) = 4x^3 - 12x$.
Are there any $x-$values that cause $f'(x)$ to not exist? $f'(x)$ is a polynomial so $f'(x)$ exists for all x.
Are there any $x-$values that cause $f'(x) = 0$?Set $f'(x) = 0$ and solve for x.

$$4x^3 - 12x = 0$$

Identify common factor $4x$ and rewrite to highlight the common factor.

$$4x \cdot x^2 - 4x \cdot 3 = 0$$

Now factor out the $4x$: $4x(x^2 - 3) = 0$

Notes

EXAMPLE 1

Step 3 Factor some more

$$4x(x - \sqrt{3})(x + \sqrt{3}) = 0$$

Solution: $x = 0, x = -\sqrt{3}, x = \sqrt{3}$ these are the partition numbers for $f'(x)$ because they cause $f'(x) = 0$.
Observe that $f(x)$ exists at all three of these partition numbers for f' (because f is a poly, so its domain is all real numbers).
So the three $x-$values $x = 0, x = -\sqrt{3}, x = \sqrt{3}$ all satisfy

- $f'(x) = 0$
- $f(x)$ exists

So these three $x-$values are the critical values for f.

Notes

EXAMPLE 1

Step 4-6 List of important $x - values$

Important $x-$values	Corresponding $y-$values
$x = -3$	$y = 32$
$x = -\sqrt{3}$	$y = -4$
$x = 0$	$y = 5$
$x = \sqrt{3}$	$y = -4$
$x = 2$	$y = -3$

Conclusion:

- The absolute max is $y = 32$ and it occurs at $x = -3$
- The absolute min is $y = -4$ and it occurs at $x = -\sqrt{3}$ and $x = \sqrt{3}$

Notes

EXAMPLE 2

Find the absolute extrema of $f(x) = x^4 - 6x^2 + 5$ on the interval $[-1, 2]$.

Step 1 The interval $[-1, 2]$ is a closed interval.

Step 2 The function f is continuous on $[-1, 2]$ because f is a polynomial

Step 3 Critical values of f:
$x = 0$ and $x = \sqrt{3}$
$\cancel{x = -\sqrt{3}}$ not in the interval $[-1, 2]$

Notes

Notes

EXAMPLE 2

Step 4-6 List of important $x - values$

Important $x-$values	Corresponding $y-$values
$x = -1$	$y = 0$
$x = 0$	$y = 5$
$x = \sqrt{3}$	$y = -4$
$x = 2$	$y = -3$

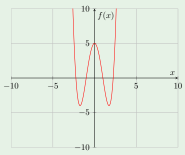
Conclusion:

- The absolute max is $y = 5$ and it occurs at $x = 0$
- The absolute min is $y = -4$ and it occurs at $x = \sqrt{3}$

Notes

EXAMPLE 3

Find the absolute extrema of $f(x) = x^4 - 6x^2 + 5$ on the interval $(-\infty, \infty)$.
Observe f is continuous but the interval is not closed. We are not guaranteed any max or min.
We cannot use the closed interval procedure!
So what do we do?
A variety of math technique have to be used, depending on the problem.
Observe f is even degree polynomial with positive leading coefficient. So both ends go up.



So graph will have absolute min, but will not have an absolute max.

Notes

EXAMPLE 3

THEOREM 1 tells us that the only places where abs max or min can occur at

- critical values
- endpoints

We don't have any endpoints in this example, so the abs max or min must occur at critical values.
From previous example, we know that the critical values of f are:
$x = 0, x = -\sqrt{3}, x = \sqrt{3}$.
So it must be that $y = -4$ is the abs min (it occurs at $x = -\sqrt{3}$ and $x = \sqrt{3}$). No abs max!

Notes

Second-Derivative Test

Let c be a critical value of $f(x)$.

$f'(c)$	$f''(c)$	Graph of f is	$f(c)$
0	$+$	Concave up	Local min
0	$-$	Concave do,wn	Local max
0	0	Concave up	Test fails

Notes

EXAMPLE 4

Find the local maximum and minimum values of $f(x) = x^3 - 6x^2$ on $[-1, 7]$.

$$f'(x) = 3x^2 - 12x = 3x(x-4)$$
$$f''(x) = 6x - 12 = 6(x-2)$$

Critical values: $x = 0$ and $x = 4$

$$f''(0) = -12, \text{hence} f(0) \text{ local max}$$
$$f''(4) = 12, \text{hence} f(4) \text{ local min}$$

Notes

THEOREM 3: **Second-Derivative Test for Absolute Extremum**

Let f be continuous on interval I with only one critical value c in I.

- If $f'(c) = 0$ and $f''(c) > 0$, then $f(c)$ is the absolute minimum of f on I.
- If $f'(c) = 0$ and $f''(c) < 0$, then $f(c)$ is the absolute maximum of f on I.

The second-derivative test does not apply if $f''(c) = 0$ or if $f''(c)$ is not defined. The first-derivative test must be used.

Notes

EXAMPLE 5

Find the absolute minimum value of $f(x) = x + \frac{4}{x}$ on $(0, \infty)$.

$$f'(x) = 1 - \frac{4}{x^2} = \frac{x^2 - 4}{x^2} = \frac{(x-2)(x+2)}{x^2}$$
$$f''(x) = \frac{8}{x^3}$$

The only critical value in the interval $(0, \infty)$ is $x = 2$. Since $f''(2) = 1 > 0$, $f(2)$ is the abs min value of f on $(0, \infty)$

Notes

EXERCISES

1. Use the second derivative test to find the local extrema for $f(x) = 2x^3 - 4x^2 - 10$
2. Let $f(x) = 20 - 4x - \frac{250}{x^2}$. Find all absolute extrema on the interval $(0, \infty)$
3. Find the absolute maxima and absolute minima, if they exist, for the function $f(x) = \frac{x^3}{3} - x^2 + 4$ on the given intervals.
   a. $[-4, 0]$
   b. $[-4, 3]$

Learning Objectives

- Solve applications requiring optimization of area or perimeter.
- Solve applications requiring optimization of revenue, profit, or cost.
- Solve inventory control applications.

Notes

Optimization involves absolute extremum problems.

Possible Complications:

- problems may be word problems.
- domain might not be closed intervals.
- domain might not even be specified (you will have to figure it out).
- problems might involve more than one variable.

The techniques used to solve optimization problems are best illustrated through examples. Let's begin with some examples.

Notes

EXAMPLE 1

Find two positive numbers x, y such that

- the product of the numbers is 9000.
- the sum $10x + 25y$ is minimized.

Two equations:

Eq I: $xy = 9000$

Eq II: $10x + 25y = S$ (minimize this Sum)

Eliminate one of the variables:
Solve equation I for y: $y = \frac{9000}{x}$
Substitute into equation II: $10x + 25\frac{9000}{x} = S$
This describes a function S of the variable x. In function notation

$$S(x) = 10x + 25\frac{9000}{x}$$

The domain is $(0, \infty)$ because x must be positive.
Goal: Find absolute min of $S(x)$ on the interval $(0, \infty)$.

Notes

EXAMPLE 1

If there are any abs extrema, we know that they can only occur at $x-$values that are critical value of $S(x)$. So we must find them.
Start by finding partition numbers of $S'(x)$ that is $x-$values where $S' = 0$ or S' DNE.

$$\begin{aligned} S(x) &= 10x + 25\frac{9000}{x} = 10x + 25(9000)x^{-1} \\ S'(x) &= \frac{d}{dx}\left(10x + 25(9000)x^{-1}\right) \\ &= 10 + 25(9000)(-1)x^{-2} \\ &= 10 - \frac{25(9000)}{x^2} \end{aligned}$$

Any $x-$values that cause S' to be undefined?
Yes: $x = 0$, but it is not in our interval $(0, \infty)$
Are there any $x-$values that cause $S'(x) = 0$?
Set $S'(x) = 0$ and solve for x.

Notes

EXAMPLE 1

$$10 - \frac{25(9000)}{x^2} = 0$$
$$10 = \frac{25(9000)}{x^2}$$
$$10x^2 = 25(9000)$$
$$x^2 = 25(900)$$
$$x = \sqrt{25(900)} = \sqrt{25}\sqrt{900}$$
$$= 5 \cdot 30 = 150$$

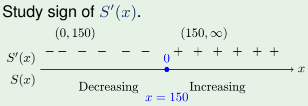
So $x = 150$ is a partition number for S' because $S'(150) = 0$.

Is $x = 150$ a critical value for S?
Does $S(150)$ exists?
$S(150) = 10(150) + \frac{25(900)}{150}$, this exists!
So $x = 150$ is a partition number for $S'(x)$ has property that $S(150)$ exists.
So $x = 150$ is a critical value for S. This must be the place where the min occurs.

Notes

EXAMPLE 1

Study sign of $S'(x)$.



So $x = 150$ is the location of the absolute min.
We still need to find y. Must satisfy

$$xy = 9000$$
$$y = \frac{9000}{x}$$
$$y = \frac{9000}{150} = 60$$

$(x, y) = (150, 60)$

Notes

EXAMPLE 2

Find the dimensions of a rectangular area of 225 square meters that has the least perimeter.

Let $L = \text{lenght}, W = \text{width}$.
The formulas for area A and perimeter P are

$$A = L \cdot W = 225$$
$$P = 2L + 2W$$

From the area equation solve for L and substitute that value of L into the perimeter equation to get an equation in one unknown:

$$L = \frac{225}{W}$$
$$P = 2\frac{225}{W} + 2W = \frac{450}{W} + 2W$$

We wish to minimize $P(W)$, so we take the derivative and look at the critical values.

Notes

EXAMPLE 2

$$P'(W) = \frac{d}{dW}\left(\frac{450}{W} + 2W\right) = \frac{-450}{W^2} + 2$$
$$= \frac{2W^2 - 450}{W^2} = \frac{2(W^2 - 225)}{W^2} = \frac{2(W-15)(W+15)}{W^2}$$

There is a critical value at $W = 15$. (Disregard $W = -15$ since the width cannot be negative).

$$P''(W) = \frac{900}{W^3}$$

$P''(15) > 0$, so this is a local minimum and since $W = 15$ is the only critical value, then $P(15) = \frac{450}{15} + 2 \cdot 15 = \$60$ must be the absolute minimum value of $P(W)$. The least perimeter occurs when $W = 15$.
For this value $L = \frac{225}{15} = 15$, so the shape is a square of side 15 meters, with minimum perimeter of 60.

Notes

PROCEDURE Strategy for Solving Optimization Problems

Step 1 Introduce variables, look for relationships among these variables, and construct a math model of the form: Maximize (minimize) $f(x)$ on the interval I.

Step 2 Find the critical values of $f(x)$.

Step 3 Find the maximum (minimum) value of $f(x)$ on the interval I.

Step 4 Use the solution to the mathematical model to answer all the questions asked in the problem.

Notes

EXAMPLE 3

A company manufactures and sells x television sets per month. The monthly cost and price-demand equations are:

$$C(x) = 60,000 + 60x$$
$$p(x) = 200ˇx/50, \qquad \text{for } 0 \leq x \leq 6,000$$

a Find the production level that will maximize the revenue, the maximum revenue, and the price that the company needs to charge at that level.

b Find the production level that will maximize the profit, the maximum profit, and the price that the company needs to charge at that level.

Notes

EXAMPLE 3

a The monthly revenue is

$$R(x) = xp(x) = x(200ˇx/50) = 200x - \frac{x^2}{50}$$

The mathematical model for this problem is

$$\text{Maximize } R(x) = 200x - \frac{x^2}{50} \qquad 0 \leq x \leq 6,000$$

Differentiate and set to zero:

$$R'(x) = 200 - \frac{x}{25} = 0$$
$$x = 5000$$

Notes

EXAMPLE 3

a Use the second-derivative test for absolute extrema:

$$R''(x) = -\frac{1}{25} < 0, \quad \text{for all} x$$

Since $x = 5000$ is the only critical value and $R''(x) < 0$,

$$\text{Max } R(x) = R(5000) = \$500,000$$

When the demand is $x = 5000$, the price is

$$p(5000) = \$100$$

b $\text{Profit} = \text{Revenue}ˇ\text{Cost}$

$$P(x) = 200x - \frac{x^2}{50} - (60000 + 60x) = -\frac{x^2}{50} + 140x - 60000$$
$$P'(x) = \frac{-x}{25} + 140 = 0$$
$$x = 3500$$

Notes

EXAMPLE 3

Use the second-derivative test for absolute extrema:

$$P''(x) = -\frac{1}{25} < 0, \quad \text{for all } x$$

Since $x = 3500$ is the only critical value and $P''(x) < 0$,

$$\text{Max } P(x) = P(3500) = \$185,000$$

When the demand is $x = 3500$, the price is

$$p(3500) = \$130$$

Summary:
The maximum revenue of $\$500,000$ is achieved at a production level of 5000 sets per month, which are sold at $\$100$ each. (The profit is $P(5000) = \$140,000$.)

The maximum profit of $\$185,000$ is achieved at a production level of 3500 sets per month, which are sold at $\$130$ each. (The revenue is $R(3500) = \$455,000$).

Notes

EXAMPLE 4: **Inventory Control**

A pharmacy has a uniform annual demand for 200 bottles of a certain antibiotic. It costs $\$5$ per year for a storage place for one bottle, and $\$40$ to place an order.

How many times during the year should the pharmacy order the antibiotic in order to minimize total cost?

Example: If you use 4 orders of 50 bottles each, you need 50 storage places. If you use 10 orders of 20 bottles each, you only need 20 storage places, but it costs more to order.

Les $x =$ number of bottles per order, and $y =$ number of orders.
The total annual cost is $C = 40y + 5x$.
In order to write the total cost C as a function of one variable, we must find a relationship between x and y.
The total number of bottles is $xy = 200$, so $y = \frac{200}{x}$.

Notes

EXAMPLE 4: **Inventory Control**

Certainly, x must be at least 1 and cannot exceed 200. We must solve the following equation:

$$\begin{aligned}
\text{Minimize } C(x) &= \frac{8000}{x} + 5x \qquad 1 \leq x \leq 200 \\
C'(x) &= -\frac{8000}{x^2} + 5 = 0 \\
x &= 40 \\
C''(x) &= \frac{8000}{x^3} > 0 \quad \text{for } x \in (1, 200)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\text{Min } C(x) = C(40) &= \frac{8000}{40} + 5 \cdot 40 = 400 \\
y &= \frac{200}{40} = 5
\end{aligned}$$

The pharmacy will minimize its total cost by ordering 40 bottles five times during the year.

Notes

EXERCISES

1. Find two positive numbers x, y such that
   - the sum $2x + y = 900$.
   - the product $A = xy$ is maximized.

2. A farmer needs to build a fence to make a rectangular yard next to an adjacent pasture. He only needs to fence 3 sides because the 4th side already has a fence. He has 900 feet of fence to use.
What dimensions give the largest yard?

3. Katie is a seamstress who makes wedding dresses. Her monthly cost and revenue functions when making x wedding dresses can be modeled approximately by $C(x) = 200 + 150x$ and $R(x) = 700x - 35x^2$, where $0 \leq x \leq 15$
   - How many dresses should Katie make each month to maximize revenue?
   - How many dresses should Katie make each month to maximize profit?
   - Are the values from parts a and b the same? If not, explain why they may be different.

Notes