Mathématiques pour SHS

Master Sciences des données et histoire

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UNIVERSITÉ PARIS 1
PANTHÉON SORBONNE

3. Limits and the Derivative

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Calculus emerged in the 17th century during the Scientific Revolution, when Isaac Newton (1642–1727) and Gottfried Wilhelm Leibniz (1646–1716), working independently in England and Germany, respectively, formulated its principles to address questions of motion (but many of its ideas can be traced back to the early days of ancient Greek mathematics).

Over time, its applications have broadened considerably, proving essential not only in physics but also across a wide spectrum of fields including business, economics, biology, and sociology,...—anywhere the study of change is relevant.

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Chap3 introduce the *derivative*, one of the two key concepts of calculus. The second, the *integral*, is the subject of Chap6. Both concepts depend on the notion of *limit*.

3. Limits and the Derivative

Learning Objectives

- Evaluate limits of function from graphs.

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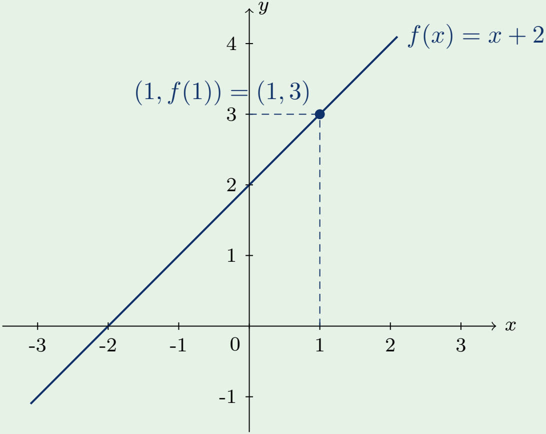
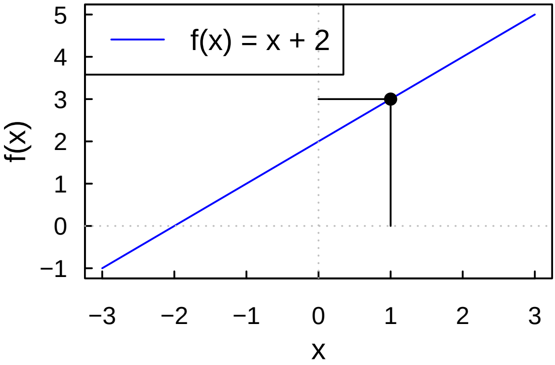
- Evaluate limits of function from graphs.
- Evaluate limits of functions algebraically.

EXAMPLE 1

The graph of the function $f(x) = x + 2$ is a visual representation of all the ordered pairs $(x, f(x))$. For instance, if $x = 2$, $(1, f(1)) = (1, 3)$, is a point on the graph of f.

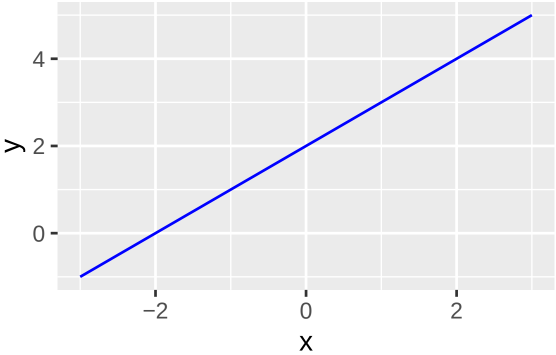


```
f <- function(x){x+2}
curve(f, from = -3, to = 3, col="blue")
abline(h=0, v=0, col="gray", lty=3)
segments(x0 = 1, y0 = 0, x1 = 1, y1 = f(1))
segments(x0 = 1, y0 = f(1), x1 = 0, y1 = f(1))
points(1, f(1), pch=16)
legend("topleft", legend="f(x) = x + 2", col="blue", lty=1)
```



```r
data_func <- data.frame(x = seq(-3, 3, by = 0.1))
data_func$y <- f(data_func$x)

p <- ggplot(data_func, aes(x = x, y = y)) +
  geom_line(color = "blue")
print(p)
```



DEFINITION **limit**

•Symbol:

$$\lim_{x \to c} f(x) = L$$

•Spoken: "The limit as x approaches c of $f(x)$ is L"

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- f is a function
- c is a real number
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•Usage:
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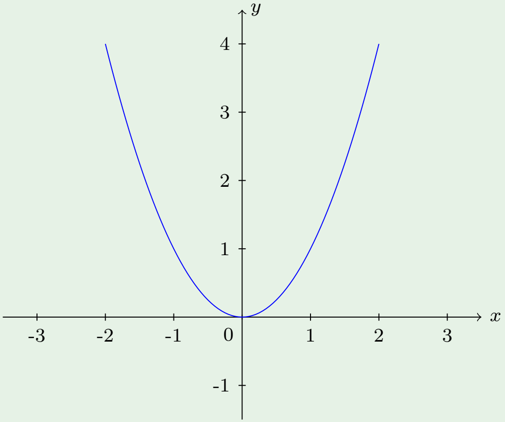
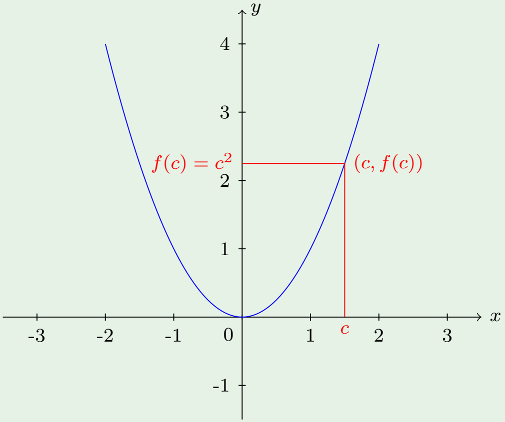
•Meaning: As x gets closer to c, but not equal to c, the values of $f(x)$ get closer and closer to L.

EXAMPLE 2



$f(x) = x^2$

EXAMPLE 2



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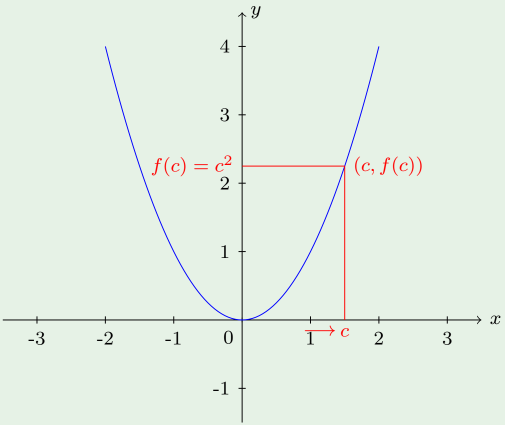

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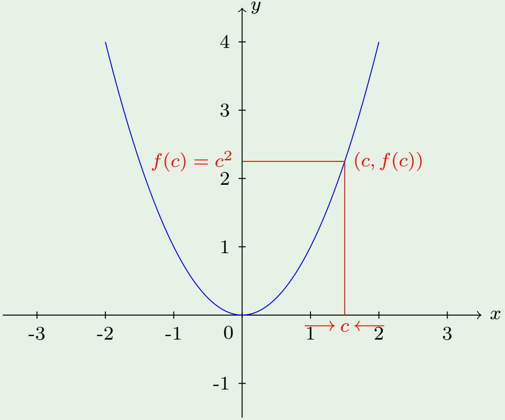

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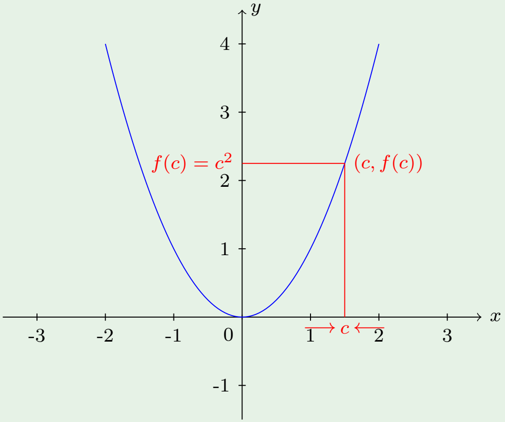

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$$\lim_{x \to c} f(x) = \lim_{x \to c} x^2 = c^2$$

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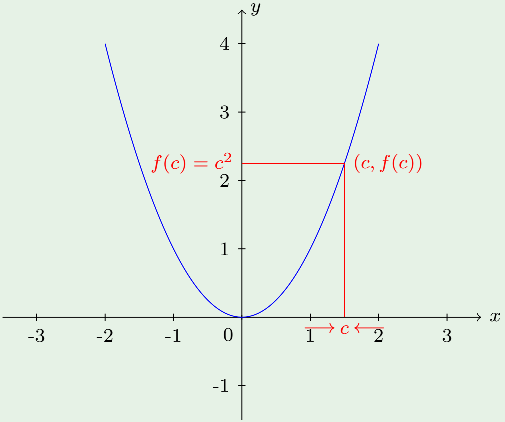

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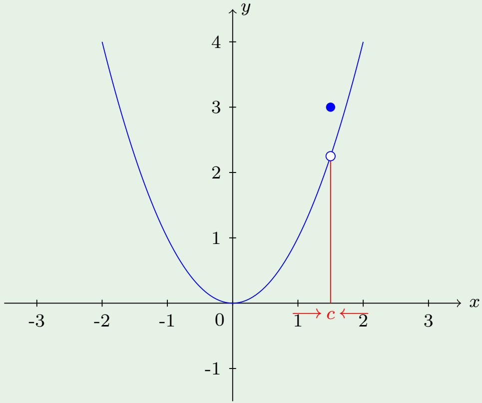

$$f(x) = \begin{cases} x^2 & \text{if } x \neq c \\ 3 & \text{if } x = c \end{cases}$$

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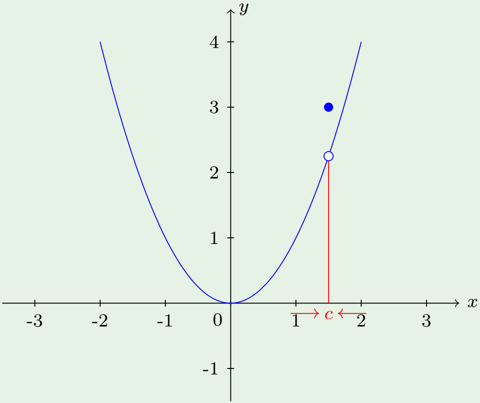

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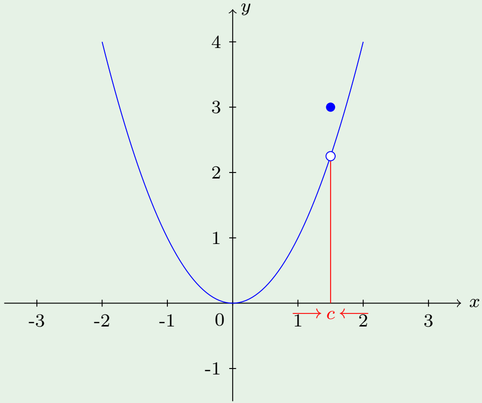

$$f(x) = \begin{cases} x^2 & \text{if } x \neq c \\ 3 & \text{if } x = c \end{cases}$$

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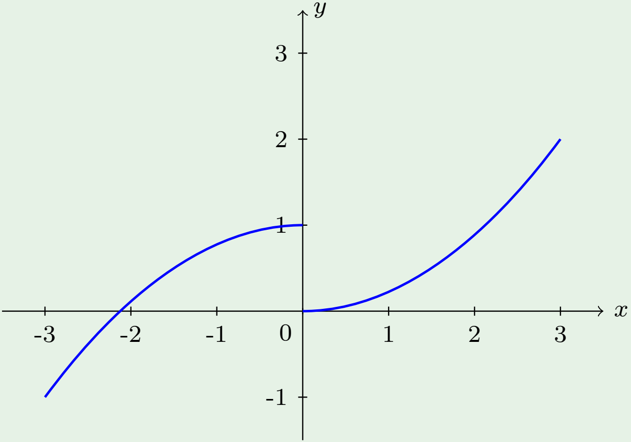
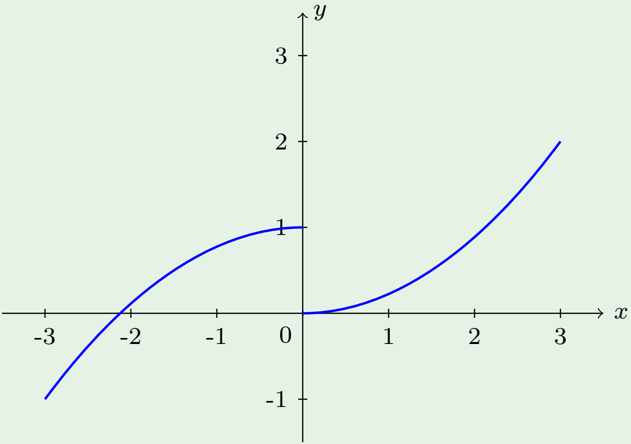
EXAMPLE 4: Situation where the limit Does Not Exist

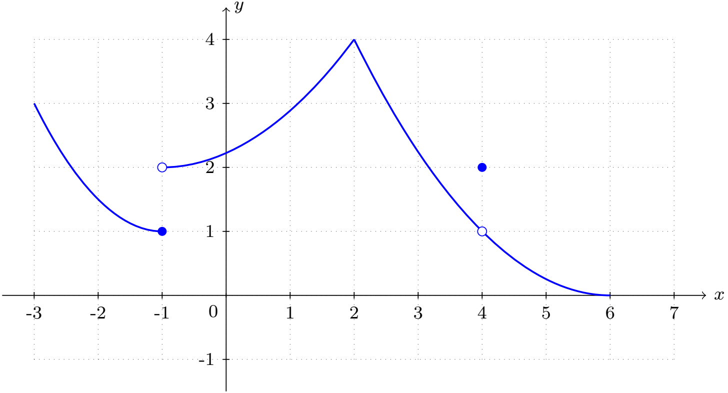


EXAMPLE 4: Situation where the limit Does Not Exist



$$\lim_{x \to 0} f(x) = \text{DNE}$$

EXERCISE: Use the graph to fill in the table



x-value	limit from left	limit from right	limit	y-value
4	$\lim\limits_{x\to 4^-} f(x) =$	$\lim\limits_{x\to 4^+} f(x) =$	$\lim\limits_{x\to 4} f(x) =$	$f(4) =$
-1	$\lim\limits_{x\to -1^-} f(x) =$	$\lim\limits_{x\to -1^+} f(x) =$	$\lim\limits_{x\to -1} f(x) =$	$f(-1) =$

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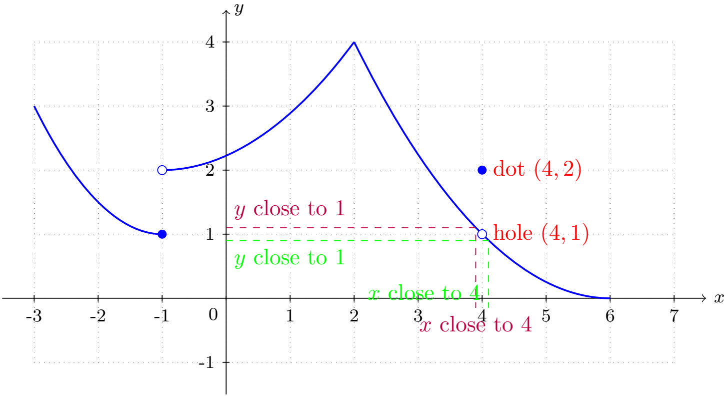

x-value	limit from left	limit from right	limit	y-value
4	$\lim_{x\to 4^-} f(x) = 1$	$\lim_{x\to 4^+} f(x) = 1$	$\lim_{x\to 4} f(x) = 1$	$f(4) = 2$
-1	$\lim_{x\to -1^-} f(x) =$	$\lim_{x\to -1^+} f(x) =$	$\lim_{x\to -1} f(x) =$	$f(-1) =$

3. Limits and the Derivative

3-1 Introduction to Limits

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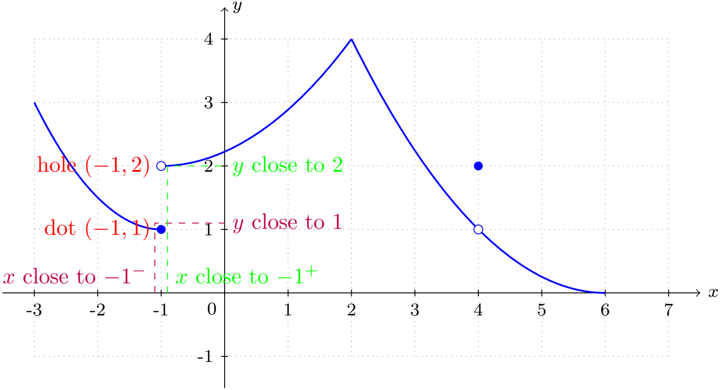

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exists.
The value of the limit is whatever was the common value of the left and right limits.

THEOREM 2 **Properties of Limits**

Let f and g be two functions, and assume that the following two limits exist and are finite:

$$\lim_{x \to c} f(x) = L, \qquad \lim_{x \to c} g(x) = M$$

Then

1. the limit of a constant is the constant

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6. the limit of the product of the functions is the product of the limits of the functions

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6. the limit of the product of the functions is the product of the limits of the functions
7. the limit of the quotient of the functions is the quotient of the limits of the functions, provided $M \neq 0$.

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6. the limit of the product of the functions is the product of the limits of the functions
7. the limit of the quotient of the functions is the quotient of the limits of the functions, provided $M \neq 0$.
8. the limit of the n^{th} root of a function is the nth root of the limit of that function

EXAMPLE 5

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$$\begin{aligned}\lim_{x\to 2} f(x) &= \lim_{x\to 2} -7x^2 + 13x - 29 \\ &= \lim_{x\to 2}(-7x^2) + \lim_{x\to 2}(13x) + \lim_{x\to 2}(-29) \quad \text{Th 2.3}\end{aligned}$$

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&= \lim_{x \to 2} (-7x^2) + \lim_{x \to 2} (13x) + \lim_{x \to 2} (-29) \quad \text{Th 2.3} \\
&= -7 \lim_{x \to 2} (x^2) + 13 \lim_{x \to 2} (x) - 29 \quad \text{Th2.5 \& Th2.1}
\end{aligned}$$

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&= -7\lim_{x\to 2}(x^2) + 13\lim_{x\to 2}(x) - 29 \quad \text{Th2.5 \& Th2.1} \\
&= -7\lim_{x\to 2}(x \times x) + 13 \times 2 - 29 \quad \text{Th2.2}
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&= -7 \lim_{x \to 2} (x \times x) + 13 \times 2 - 29 \quad \text{Th2.2} \\
&= -7 \left(\lim_{x \to 2} (x)\right) \left(\lim_{x \to 2} (x)\right) + 26 - 29 \quad \text{Th2.6}
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&= -7\left(\lim_{x\to 2}(x)\right)\left(\lim_{x\to 2}(x)\right) + 26 - 29 \quad \text{Th2.6} \\
&= -7(2)(2) + 26 - 29 \quad \text{Th2.1 again}
\end{aligned}$$

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&= -31
\end{aligned}$$

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Alternate solution

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} -7x^2 + 13x - 29 \quad \text{notice: } f \text{ is a polynomial}$$

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$$\begin{aligned}\lim_{x\to 2} f(x) &= \lim_{x\to 2} -7x^2 + 13x - 29 \quad \text{notice: } f \text{ is a polynomial} \\ &= -7(2)^2 + 13(2) - 29 \quad \text{can just substitute in } x = 2 \text{ by using Th3}\end{aligned}$$

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From this example, we conclude that

THEOREM 3 **Limits of Polynomial and Rational Functions**

$$\lim_{x \to c} f(x) = f(c) \quad \text{for } f \text{ any polynomial function}$$

To find the limit of a polynomial, just plug-in the value!

EXAMPLE 6

Let $r(x) = \frac{2x}{3x+1}$, find $\lim_{x \to 4} r(x)$

$$\lim_{x \to 4} r(x) = \lim_{x \to 4} \frac{2x}{3x+1}$$

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From this example, we conclude that

THEOREM 3 **Limits of Polynomial and Rational Functions**

$$\lim_{x\to c} r(x) = r(c)$$ r any rational function with a nonzero denominator at $x = c$

Indeterminate Forms

It is important to note that there are restrictions on some of the limit properties. In particular if

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Evaluate the following limit:

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Algebraic simplification is often useful when the numerator and denominator are both approaching 0

THEOREM 4 Limit of a Quotient

If $\lim_{x\to c} f(x) = L, L \neq 0$, and $\lim_{x\to c} g(x) = 0$, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} \quad \text{does not exist}$$

Difference Quotient

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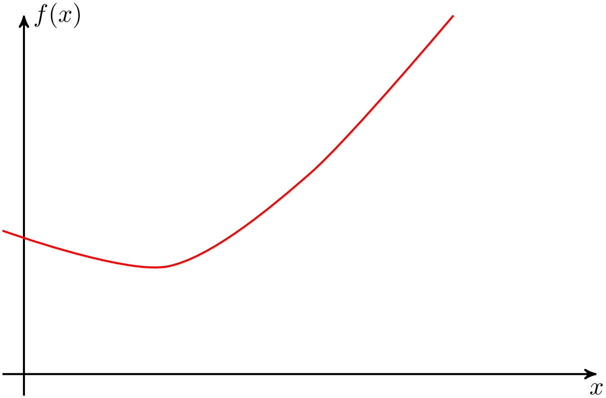

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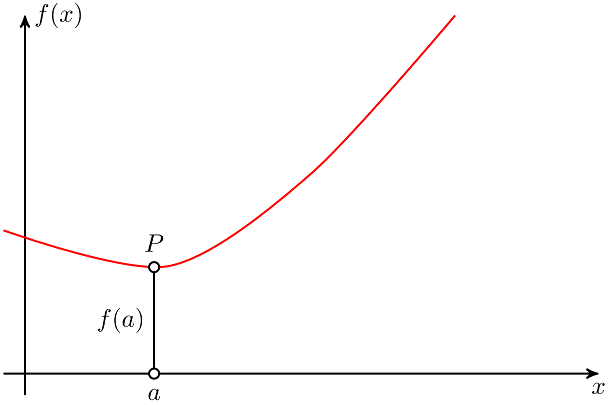

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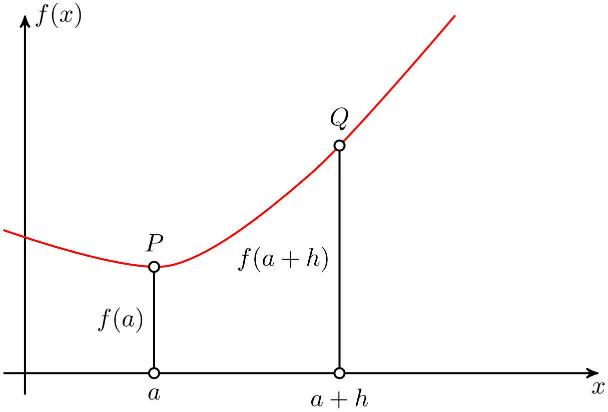

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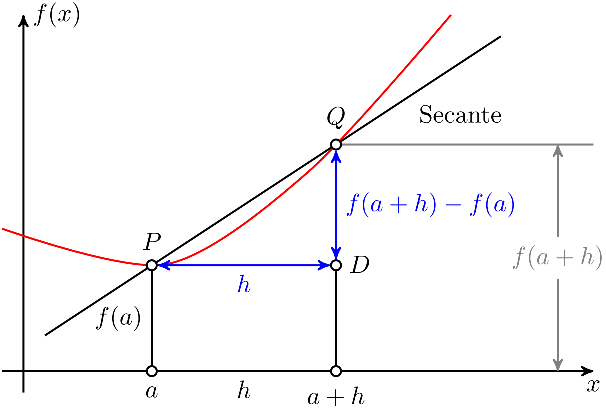

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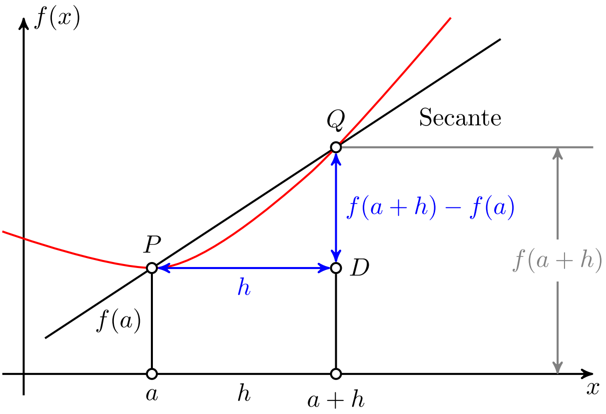

$$\text{Slope of } PQ = \frac{QD}{PD} = \frac{f(a+h)-f(a)}{(a+h)-a} = \frac{f(a+h)-f(a)}{h}$$

EXAMPLE 8

Let $f(x) = 3x - 1$, find $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$

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$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{3h}{h} = 3$$

EXERCISES

1. Evaluate the following limits.

a

$$\lim_{x \to 3} \frac{x-2}{x-3}$$

b

$$\lim_{x \to -3} \frac{x^2-9}{x-3}$$

c

$$\lim_{x \to 1} \frac{x^2+6x-7}{x-1}$$

2. Compute the following limits for each function: $\lim_{h \to 0} \frac{f(1+h)-f(1)}{h}$

a $f(x) = 4x + 3$

b $f(x) = x^2 - 2x - 5$

c $f(x) = \sqrt{x+3}$

EXERCICE: Carbon Tax I

France applies a carbon tax to incentivize the reduction of greenhouse gas emissions. The tax rate in 2021 was approximately EUR 44 per ton of CO2. Assume for this exercise that the tax applies to all emissions without a cap, reflecting a policy aimed at full-cost internalization.

1. Write a piecewise definition of the fees $F(x)$ charged for the emission of x tons of CO2 in a year. Consider a scenario where after a certain threshold, say $5,000$ tons, the tax rate increases to encourage industrial-scale emitters to invest in cleaner technologies.
2. What is the limit of $F(x)$ as x approaches the threshold?
3. And as x approaches a much higher value, indicating the practical ceiling for emissions for the largest polluters?

EXERCICE: Carbon Tax II

Referring to the pollution tax policy from the previous exercise, consider that the French government is contemplating a new tiered fee system to further penalize higher emissions. Under this system, a company is charged a base rate for emissions up to a specified limit and a higher rate beyond that limit, to an upper limit after which the rate does not increase.

1. Write a piecewise function for the tiered tax rate $A(x)$ that reflects the following hypothetical structure:
   1. EUR 44 per ton up to 2,000 tons of CO2 emissions,
   2. EUR 55 per ton for emissions between 2,000 and 5,000 tons,
   3. A fixed fee for emissions above 5,000 tons, reflecting the maximum tax cap applied.
2. Determine the behavior of $A(x)$ as x approaches the first and second limits. What happens as x greatly exceeds the second limit?

3. Limits and the Derivative

Learning Objectives

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- Locate horizontal asymptotes.

In Section 3.1, we introduced the expression

$$\lim_{x \to c} f(x) = L$$

Spoken: "The limit as x approaches c of $f(x)$ is L"
Meaning: The graph of f appears to be heading for the location $(x, y) = (c, L)$.

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DEFINITION Infinite Limits and Vertical Asymptotes

The vertical line $x = a$ is a vertical asymptote for the graph of $y = f(x)$ if

$$f(x) \to \infty \quad \text{or} \quad f(x) \to -\infty \quad \text{as} \quad x \to a^+ \quad \text{or} \quad x \to a^-$$

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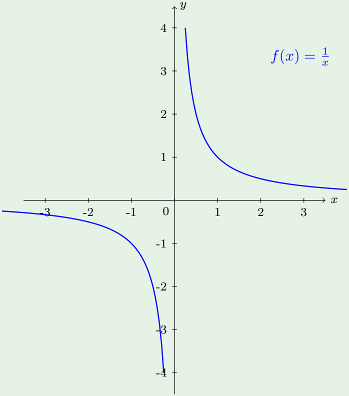
Infinite limits and vertical asymptotes are used to describe the behavior of functions that are unbounded near $x = a$.

EXAMPLE 1



What are the left and right limits of f at 0?

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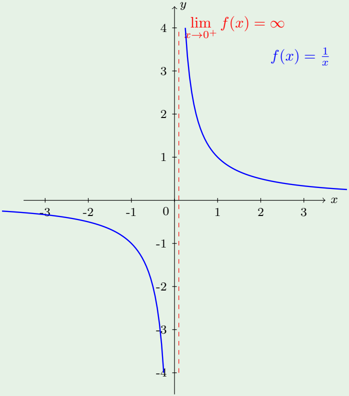

As x gets closer and closer to 0 from the right but not equal to 0 the y-values go to ∞.

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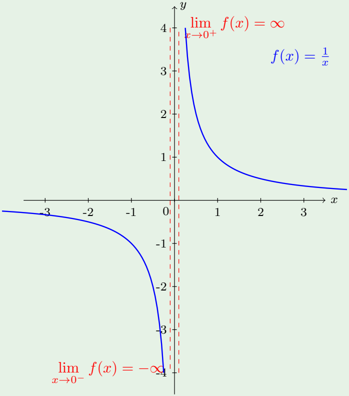

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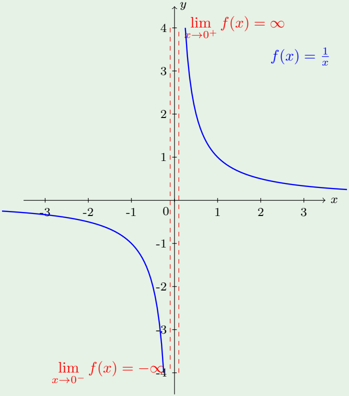
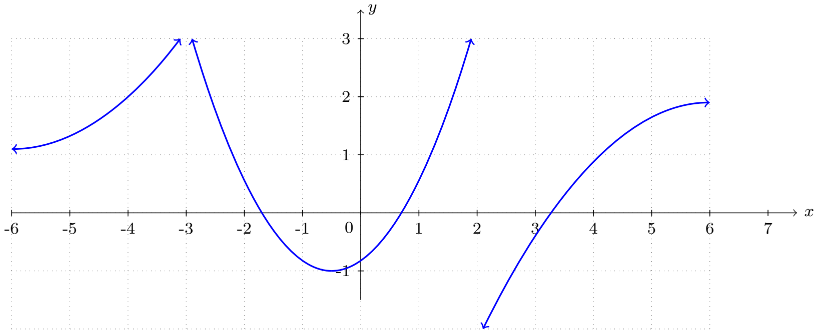


As x gets closer and closer to 0 from the left but not equal to 0 the y-values go to $-\infty$.

$$\lim_{x\to 0^+} f(x) = +\infty, \quad \lim_{x\to 0^-} f(x) = -\infty, \quad \lim_{x\to 0} f(x) = DNE$$

EXERCISE: Use the graph to fill in the table



x-value	limit from left	limit from right	limit
-3	$\lim\limits_{x \to -3^-} f(x) =$	$\lim\limits_{x \to -3^+} f(x) =$	$\lim\limits_{x \to -3} f(x) =$
2	$\lim\limits_{x \to 2^-} f(x) =$	$\lim\limits_{x \to 2^+} f(x) =$	$\lim\limits_{x \to 2} f(x) =$

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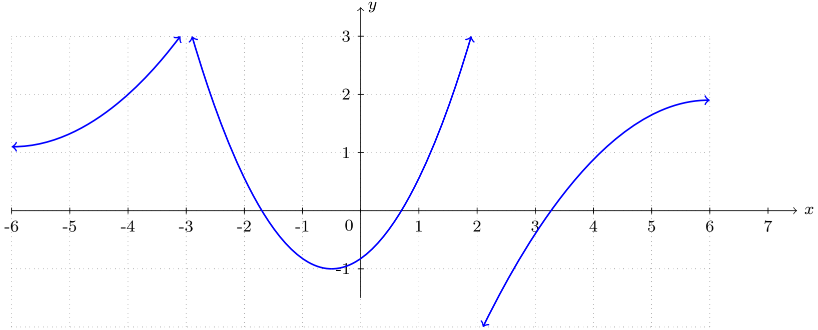

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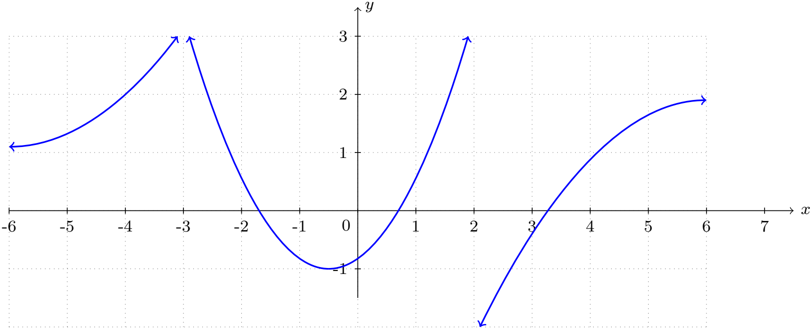

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THEOREM1 **Locating Vertical Asymptotes of Rational Functions**

If $f(x) = \frac{n(x)}{d(x)}$ is a rational function, $d(c) = 0$ and $n(c) \neq 0$, then the line $x = c$ is a vertical asymptote of the graph of f.

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If $f(x) = \frac{n(x)}{d(x)}$ is a rational function, $d(c) = 0$ and $n(c) \neq 0$, then the line $x = c$ is a vertical asymptote of the graph of f.

In other words

Vertical asymptotes occur for those value of x that produce 0 in the denominator BUT NOT in the numerator. (If $\frac{0}{0}$ occurs, you simply have a hole in the graph).

EXERCISE:

Find any vertical asymptotes :

$$a)\ f(x) = \frac{5}{x^2 - 9}, \quad b)\ g(x) = \frac{x-1}{x-4}, \quad c)\ h(x) = \frac{x-2}{x^2 + x - 6}$$

Limits at Infinity

Limits at infinity and horizontal asymptotes are used to describe the behavior of functions as x assumes arbitrarily large positive values or arbitrarily large negative values.

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We begin by considering power functions of the form x^p and $\frac{1}{x^p}$ If p is a positive real number, then x^p increases as x increases. There is no upper bound on the values of x^p. We indicate this behavior by writing

$$\lim_{x \to \infty} x^p = \infty.$$

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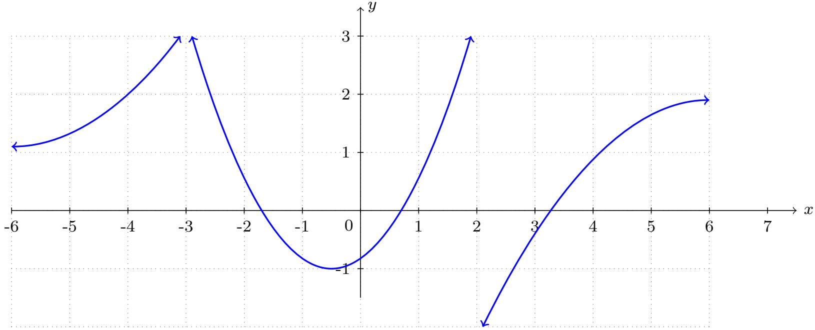
Since the reciprocals of very large numbers are very small numbers, it follows that $\frac{1}{x^p}$ approaches 0 as x increases without bound. We indicate this behavior by writing

$$\lim_{x \to \infty} \frac{1}{x^p} = 0$$

EXERCISE: Use the graph to fill in the table



x-value	limit
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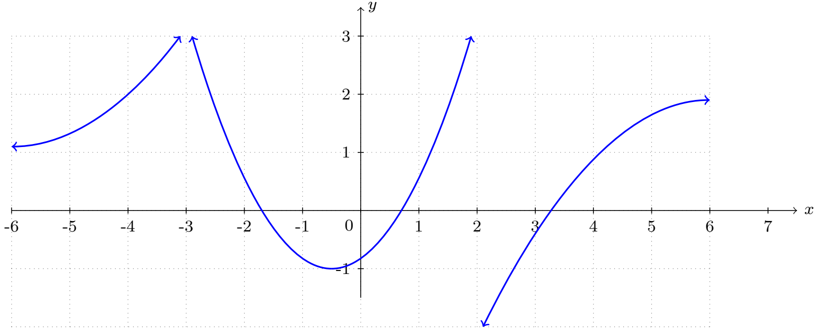

x-value	limit
∞	$\lim_{x\to\infty} f(x) = 2$
$-\infty$	$\lim_{x\to-\infty} f(x) = 1$

THEOREM 2 **Limits of Power Functions at Infinity**

If p is a positive real number and k is any real number except 0, then

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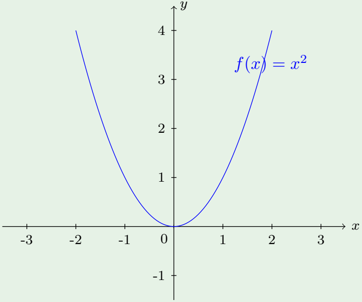
EXAMPLE 2: $f(x) = x^2$



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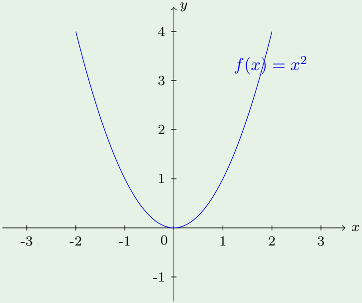
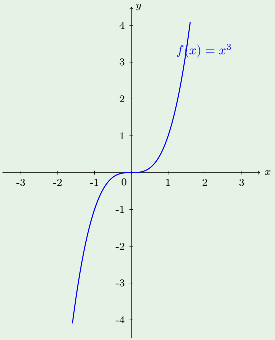


$k = 1 > 0, p = 2$ (even)

$$\lim_{x \to +\infty} x^2 = \infty, \quad \lim_{x \to -\infty} x^2 = \infty$$

EXAMPLE 3: $f(x) = x^3$



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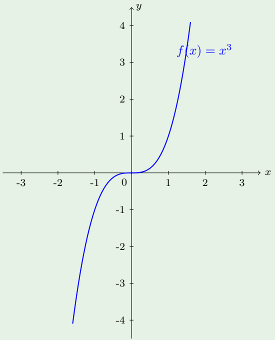
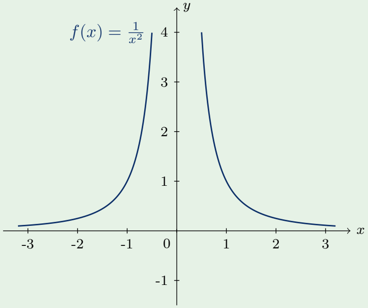


$k = 1 > 0, p = 3$ (odd)

$$\lim_{x \to +\infty} x^3 = \infty, \quad \lim_{x \to -\infty} x^3 = -\infty$$

EXAMPLE 4: $f(x) = x^{-2}$



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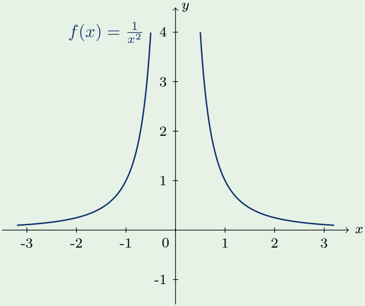
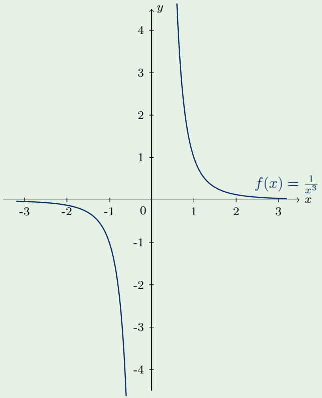


$k = 1 > 0, p = 2$ (even)

$$\lim_{x \to +\infty} \frac{1}{x^2} = 0, \quad \lim_{x \to -\infty} \frac{1}{x^2} = 0$$

EXAMPLE 5: $f(x) = x^{-3}$



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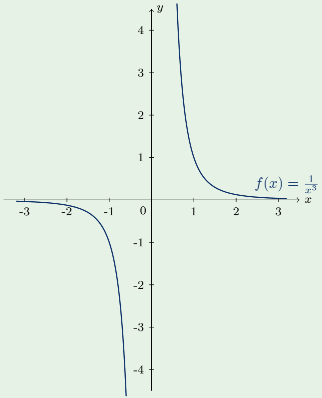

$k = 1 > 0, p = 3$ (odd)

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THEOREM 3 **Limits of Polynomial Functions at Infinity**

If

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x^1 + a_0, \quad a_n \neq 0, n \geq 1,$$

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and

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Each limit will be either $-\infty$ or ∞, depending on a_n and n.

EXAMPLE 6: $p(x) = 2x^5 + 4x^3 + 7x^2 + 4$

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$$\lim_{x \to +\infty} p(x) = \lim_{x \to +\infty} 2x^5 + 4x^3 + 7x^2 + 4 = \infty$$

EXAMPLE 7: $q(x) = 2x^5$

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$$\lim_{x \to +\infty} q(x) = \lim_{x \to +\infty} 2x^5 = \infty$$

THEOREM 4 Limits of Rational Functions at Infinity and Horizontal Asymptotes of Rational Functions

1. If

$$f(x) = \frac{a_m x^m + a_{m-1}x^{m-1} + \ldots + a_1 x^1 + a_0}{b_n x^n + b_{n-1}x^{n-1} + \ldots + b_1 x^1 + b_0}, \quad a_m \neq 0, b_n \neq 0$$

then

$$\lim_{x\to\infty} f(x) = \frac{a_m x^m}{b_n x^n}, \quad \text{and} \quad \lim_{x\to-\infty} f(x) = \frac{a_m x^m}{b_n x^n}$$

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2. There are three possible cases for these limits:
   a) if $m < n$, then $\lim_{x \to \infty} f(x) = \lim_{x \to -\infty} f(x) = 0$ and the line $y = 0$ (the x axis) is a horizontal asymptote of $f(x)$.

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   c) if $m>n$, then each limit will be $\pm\infty$, depending on m, n, a_m and b_n, and $f(x)$ does not have a horizontal asymptote

EXERCISES:

1. Find each limit for $f(x) = \frac{x-1}{x+2}$.
   - a $\lim_{x\to -2^+} f(x)$
   - b $\lim_{x\to -2^-} f(x)$
   - c $\lim_{x\to -2} f(x)$
2. Evaluate the indicated limit
   - a $\lim_{x\to\infty} \frac{x+3}{2x-1}$
   - b $\lim_{x\to\infty} \frac{4x^3+2x}{x^2-1}$
3. Discuss three different functions f, g, h

$$a)f(x) = \frac{9x^2 - 90x + 189}{2x^2 - 24x + 70},\ \ b)g(x) = \frac{9x^2 - 90x + 189}{2x^3 - 24x^2 + 70x},\ \ c)h(x) = \frac{9x^3 - 90x^2 + 189x}{2x^2 - 24x + 70}$$

   - a Would the graph of f, g, h have horizontal asymptotes?
   - b How can we find out without drawing the graph?
   - c Answer by taking the limits as $x \to \infty$ or $x \to -\infty$.

3. Limits and the Derivative

Learning Objectives

- Use the definition of continuity to determine if a function is continuous.

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- Use continuity properties to determine intervals of continuity for symbolic functions.

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- Construct sign charts to solve inequalities.

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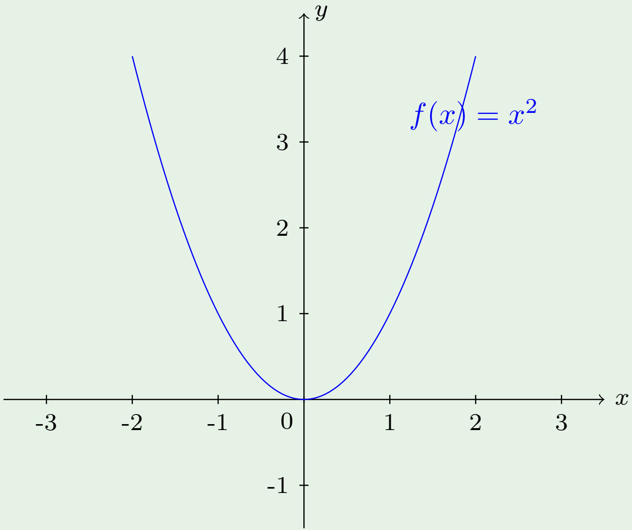
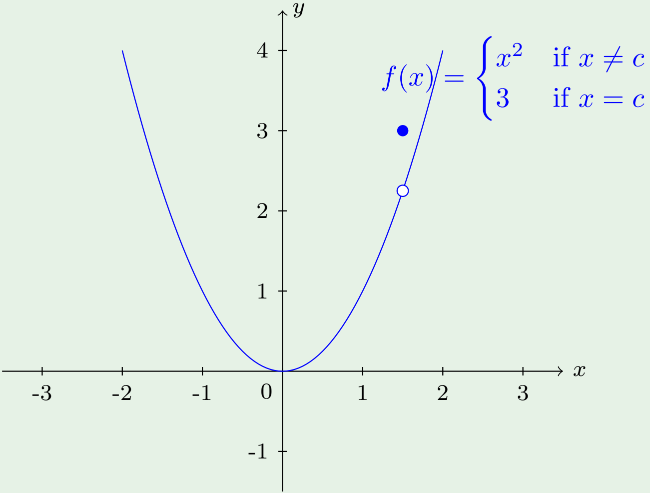
A function is continuous on the open interval (a, b) if it is continuous at each point on the interval.

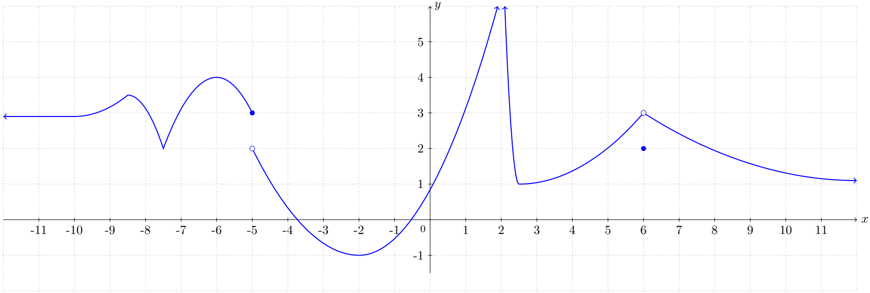
EXAMPLE 1



EXAMPLE 2


y
x
4
3
2
1
-1
-3
-2
-1
0
1
2
3
$f(x) = \begin{cases} x^2 & \text{if } x \neq c \\ 3 & \text{if } x = c \end{cases}$

EXERCISE: Use the graph to answer the questions that follow



1. For each asymptote, give the line equation and say whether it is horizontal/vertical.
2. $\lim_{x\to-\infty} f(x)$, $\lim_{x\to-5} f(x)$, $\lim_{x\to-2} f(x)$, $\lim_{x\to 6} f(x)$, $\lim_{x\to\infty} f(x)$
3. If f continuous at $a=-5$? If not, explain why not.
4. If f continuous at $a=-2$? If not, explain why not.
5. If f continuous at $a=2$? If not, explain why not.
6. If f continuous at $a=6$? If not, explain why not.

EXERCISE: Discuss the continuity of each function at the indicated point

1.
$$f(x) = x + 1, \quad \text{at} \quad x = 10$$

2.
$$g(x) = \frac{x^2 - 9}{x - 3}, \quad \text{at} \quad x = 3$$

3.
$$h(x) = \frac{|x - 1|}{x - 1}, \quad \text{at} \quad x = 1, \quad \text{and} \quad x = 0$$

If two fun are continuous on the same interval, then their $\oplus$, $\ominus$, $\otimes$, and $\oslash$ are continuous on the same interval except for values of x that make a denominator 0.

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   $f(x) = 2$ is continuous for all x.

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5. For n an odd positive integer greater than 1, $\sqrt[n]{f(x)}$ is continuous wherever $f(x)$ is continuous.
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6. For n an even positive integer, $\sqrt[n]{f(x)}$ is continuous wherever $f(x)$ is continuous and non negative.
   $\sqrt[4]{x}$ is continuous on the interval $[0, \infty)$.

EXERCISE: Determine where each function is continuous

1. $$f(x) = x^{1024} + 3x^2 + 1$$

2. $$g(x) = \frac{x^2}{(x+1)(x-8)(x+3)}$$

3. $$h(x) = \sqrt[5]{x^2 - 1}$$

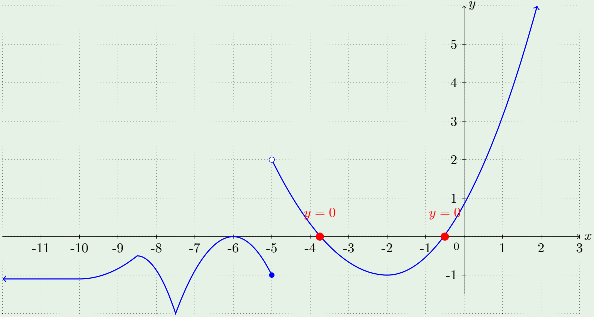
4. $$l(x) = \sqrt{x - 4}$$

Remark : Notice important behavior of functions

Common question: for some function $f(x)$,

- which $x-$values will have positive $y-$values?
- which $x-$values will have negative $y-$values?
- which $x-$values will have $y = 0$?

Observation: The sign of a function f can only change at $x-$values such that $f(x) = 0$ or f is discontinuous at x.



In general, if f is continuous and $f(x) \neq 0$ on the interval (a, b), then $f(x)$ cannot change sign on (a, b).

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THEOREM 2 Sign Properties on an Interval (a, b)

If f is continuous on (a, b) and $f(x) \neq 0$ for all x in (a, b),
then,

1. either $f(x) > 0$ for all x in (a, b)
2. or $f(x) < 0$ for all x in (a, b).

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Th 2 provides the basis for an effective method of solving many types of inequalities.

PROCEDURE Constructing Sign Charts

Given a function f,

Step 1 Find all partition numbers:

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Step 3 Select a test number in each open interval determined in step 2 and evaluate $f(x)$ at each test number to determine whether $f(x)$ is positive or negative in each interval.

Step 4 Construct a sign chart, using the real-number line in step 2.This will show the sign of $f(x)$ on each open interval.

EXERCISE (see EXAMPLE 4 p 159)

1. Solve

$$\frac{x+1}{x-2} > 0.$$

3. Limits and the Derivative

Learning Objectives

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- Identify locations of nonexistence of the derivative.

DEFINITION Average Rate of Change

For $y = f(x)$, the average rate of change from $x = a$ to $x = a + h$ is

$$\frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}, \quad h \neq 0$$

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See the picture on the next slide for illustration.

Average Rate of Change, difference quotient, slope of a secant

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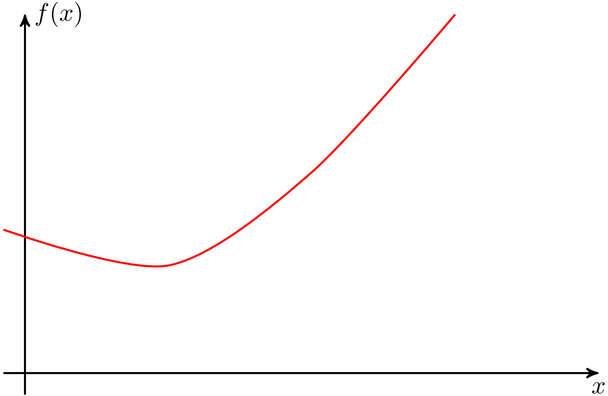
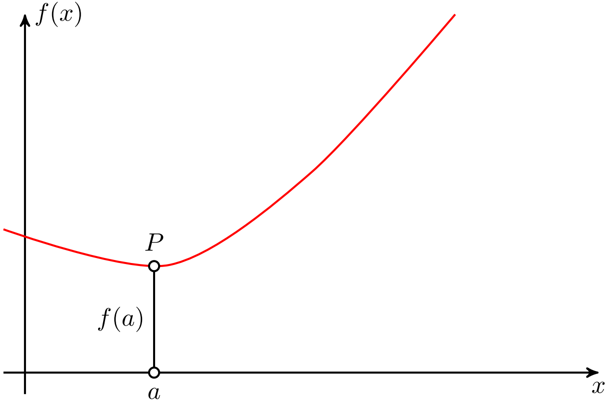


Average Rate of Change, difference quotient, slope of a secant



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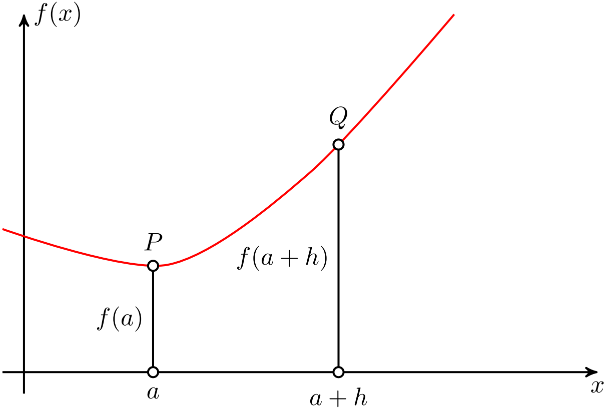

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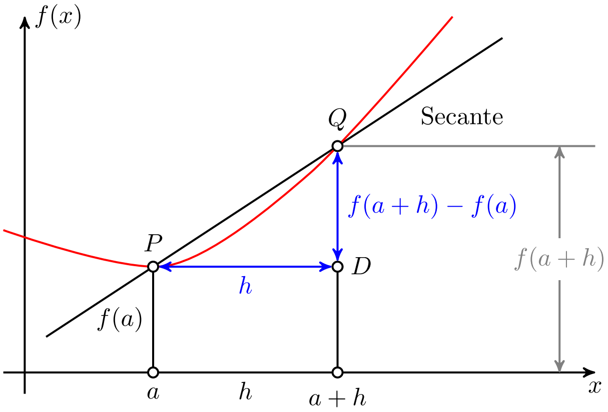

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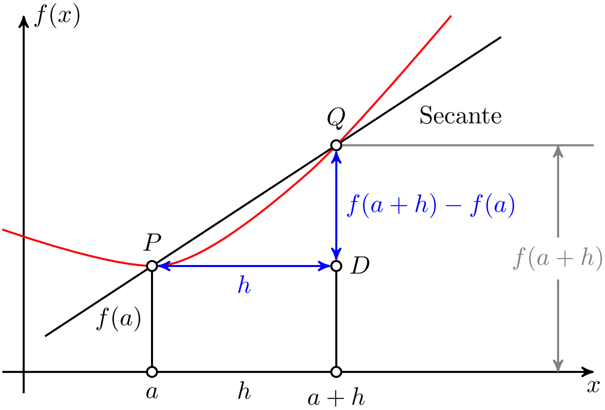

$$\text{Slope of } PQ = \frac{QD}{PD} = \frac{f(a+h)-f(a)}{(a+h)-a} = \frac{f(a+h)-f(a)}{h}$$

EXAMPLE 1

The revenue generated by producing and selling widgets is given by

$$R(x) = x(75 - 3x) \quad \text{for} \quad 0 \leq x \leq 20$$

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Increasing production from 9 to 12 will increase revenue by $36.
What is the average rate of change in revenue (per unit change in x) if production changes from 9 to 12?
To find the average rate of change we divide the change in revenue by the change in production:

$$\frac{R(12) - R(9)}{12 - 9} = \frac{36}{3} = 12$$

Thus the average change in revenue is $12 when production is increased from 9 to 12.

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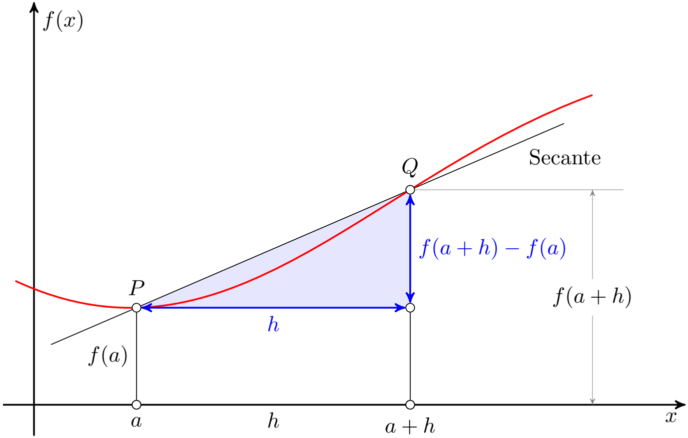
It can be interpreted as the slope of the tangent at the point P $(a, f(a))$.
See illustration on the next slide.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if the limit exists.

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- If $f'(a)$ exists, we call f differentiable at a.
- If $f'(x)$ exists for each x in the open interval (a, b), then f is said to be differentiable over (a, b).

Interpretation of Derivative

If f is a function, then f' is a new function with the following interpretations:

1. For each x in the domain of f', $f'(x)$ is the slope of the line tangent to the graph of f at the point $(x, f(x))$.

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2. For each x in the domain of f', $f'(x)$ is the instantaneous Rate of Change of $y = f(x)$ with respect to x.
3. If $f(x)$ is the position of a moving object at time x, then $v = f'(x)$ is the velocity of the object at that time.

PROCEDURE Finding the Derivative

To find $f'(x)$, we use a four-step process:

Step 1 find $f(x+h)$

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EXAMPLE 3

Find the slope of the tangent to the graph of $f(x) = x^2 - 3x$ at $x = 0, x = 2$ and $x = 3$.

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Find the slope of the tangent to the graph of $f(x) = x^2 - 3x$ at $x = 0, x = 2$ and $x = 3$.

In Example 2, we found the derivative of this function at x to be $f'(x) = 2x - 3$.
Hence,

$$f'(0) = -3$$
$$f'(2) = 1$$
$$f'3) = 3$$

EXERCISE

Find the derivative of $f(x) = 2x - 3x^2$ using the four step process.

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Non Existence of the Derivative

The existence of a derivative at $x = a$ depends on the existence of the limit

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If the limit DNE, we say that the function is nondifferentiable at $x = a$, or $f'(a)$ DNE.

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- The graph of f has a vertical tangent at $x = a$.

EXERCISE

1. Suppose a manufacturer's monthly profit (in dollars) from the sale of x bags of a particular Bermuda grass fertilizer is given by $P(x) = -0.07x^2 + 70x - 100$, where $2 \leq x \leq 998$.
   a. Find the average rate of change of profit if production is changed from 100 bags of fertilizer monthly to 500 bags of fertilizer monthly.
   b. Explain the meaning of the value obtained in part a in the context of the problem.
   c. Find the instantaneous rate of change of profit when 200 bags of fertilizer are sold. Explain the meaning of this value in the context of the problem.
2. Use the four-step process to find $f'(x)$ for $f(x) = x + 3x^2$.

3. Limits and the Derivative

Learning Objectives

- Calculate the derivative of a constant function.

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- Calculate the derivative of a constant function.
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- Apply the constant multiple and sum and difference properties.

NOTATION The Derivative

Notation for the derivative of a function, if $y = f(x)$, then

without variable f', y', $\frac{dy}{dx}$

with variable $f'(x)$, $\dfrac{d}{dx}f(x)$

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All represent the derivative of f at x.

What is the slope of a constant function?

3. Limits and the Derivative

3-5 Basic Differentiation Properties

What is the slope of a constant function?
Consider graph of $f(x) = 1$ and consider slopes of tangent lines.

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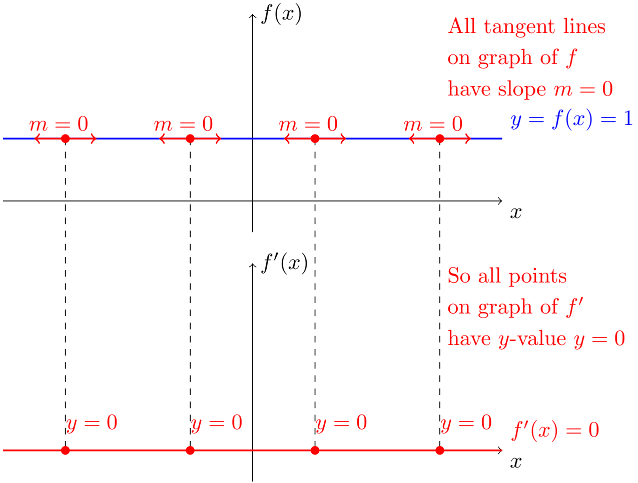

THEOREM 1 Constant Function Rule

Let $y = f(x) = C$ be a constant function, then

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THEOREM 2 **The Power Rule** (IT WILL BE USED A LOT!)

If $f(x) = x^n$, then

$$f'(x) = nx^{n-1}$$

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EXAMPLE 4

$f(x) = 3^x$, find $f'(x)$.
Not a power function. Power rule does not apply!! Can't do it (this week)

EXAMPLE 5

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$f(x) = \sqrt[5]{x}$, find $f'(x)$.
Must rewrite f as a power function $f(x) = \sqrt[5]{x} = x^{\frac{1}{5}}$.
$f'(x) = \left(\frac{1}{5}\right) x^{\frac{1}{5}-1} = \left(\frac{1}{5}\right) x^{\frac{-4}{5}} = \frac{1}{5x^{\frac{4}{5}}}$

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EXAMPLE 7

$f(x) = x$, find $f'(x)$.
Write f as a power function $f(x) = x^1$.
$f'(x) = 1x^{1-1} = 1 \cdot x^0 = 1 \cdot 1 = 1 \quad \left(\frac{dx}{dx} = 1\right)$

Does this make sense graphically?

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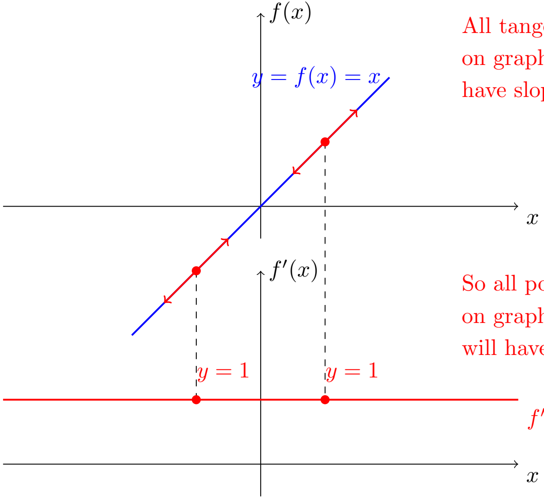
Consider graph of $f(x) = x$ and find $f'(x)$ graphically.



All tangent lines on graph of f have slope $m = 1$

So all points on graph of f' will have $y = 1$

THEOREM 3 Constant Multiple Property

If $y = f(x) = ku(x)$, then

$$f'(x) = ku'(x)$$

$$(y' = ku', \quad \frac{dy}{dx} = k\frac{du}{dx})$$

The derivative of a constant $\times$ a differentiable function is the constant $\times$ the derivative of the function.

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THEOREM 3 Sum and Difference Property

If $y = f(x) = u(x) \pm v(x)$, then

$$f'(x) = u'(x) \pm v'(x)$$

The derivative of the $\pm$ of two differentiable functions is the $\pm$ of the derivatives of the functions.

Remark: The Sum and Constant Multiple Rule

$$\frac{d}{dx}\left(af(x)+bg(x)\right)=a\frac{d}{dx}f(x)+b\frac{d}{dx}g(x)=af'(x)+bg'(x)$$

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EXAMPLE 8

$f(x)=-3x^2+5x-7$, find $f'(x)$

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EXAMPLE 8

$f(x)=-3x^2+5x-7$, find $f'(x)$

$$f'(x)=\frac{d}{dx}\left(-3x^2+5x-7(1)\right)$$ Identify multiplicative contants

$$=-3\frac{d}{dx}x^2+5\frac{d}{dx}x-7\frac{d}{dx}1$$ Sum and Constant Multipe Rule

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$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(-3x^2+5x-7(1)\right) && \text{Identify multiplicative contants}\\ &= -3\frac{d}{dx}x^2+5\frac{d}{dx}x-7\frac{d}{dx}1 && \text{Sum and Constant Multipe Rule}\\ &= -3(2x^{2-1})+5(1)-7(0) && \text{Power Rule} \end{aligned}$$

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EXAMPLE 8

$f(x) = -3x^2 + 5x - 7$, find $f'(x)$

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(-3x^2 + 5x - 7(1)\right) && \text{Identify multiplicative contants} \\ &= -3\frac{d}{dx}x^2 + 5\frac{d}{dx}x - 7\frac{d}{dx}1 && \text{Sum and Constant Multipe Rule} \\ &= -3(2x^{2-1}) + 5(1) - 7(0) && \text{Power Rule} \\ &= -6x + 5 \end{aligned}$$

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$f(x) = 3 - \frac{5}{x}$, find $f'(x)$

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Rewrite f as constants$\cdot$ power function

$$f(x) = 3 - \frac{5}{x} = 3 - 5\left(\frac{1}{x}\right) = 3 - x^{-1}$$

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Now take the derivative

$$f'(x) = \frac{d}{dx}\left(3 - x^{-1}\right)$$

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EXERCISE

a. $f(x) = 3 - 5\sqrt{5}$, find $f'(x)$.

b. $f(x) = \frac{23x^3}{19} - \frac{7}{35x^5}$, find $f'(x)$.

c. $f(x) = \frac{2\sqrt[7]{x}}{13} - \frac{3}{17x^{\frac{2}{5}}}$, find $f'(x)$.

Applications

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- Marginal Cost:

  If $C(x)$ is the cost function, that is, the total cost of producing x items, then $C'(x)$ approximates the cost of producing one more item at a production level of x items. $C'(x)$ is called the marginal cost.

Application Example: Tangent Line Problems

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- The tangent line has slope $m = f'(a)$

Recall: "point slope" form of the equation of a line

If we know that a line

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Build the equation for the line

$$y - f(a) = f'(a)(x - a)$$ equation of the tangent line

EXAMPLE 10

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$a = 2$ ($x-$coord of point of tangency)
$f(2) = 2^3 - 9 \cdot 2^2 + 15 \cdot 2 + 25 = 8 - 36 + 30 + 25 = 27$ ($y-$coord of point of tangency)

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$f'(a) = f'(2) = 3(2)^2 - 18(2) + 15 = -9$ (slope of the tangent line) Substitute the parts into the equation:

$$y - 27 = (-9)(x - 2) \quad \text{point slope form of the equation}$$

Convert to slope intercept form

$$\begin{aligned} y - 27 &= (-9)(x - 2) \\ y &= -9x + 18 + 27 \\ y &= -9x + 45 \quad \text{slope intercept form of the equation} \end{aligned}$$

Application Example: Marginal Cost

The total cost of producing x laptop per day is

$$C(x) = 1000 + 100x - 0.5x^2, \quad \text{for} \quad 0 \leq x \leq 100$$

1. Find the marginal cost of production at a production level of x laptops.
2. Find the marginal cost of production at a production level of 80 laptops.
3. Find the actual cost of producing the 81$^{\text{st}}$ laptop and compare this with the marginal cost.

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3. The actual cost of the 81st laptop will be $C(81) - C(80) = \$5819.50 - \$5800 = \$19.50$. This is approximately equal to the marginal cost.

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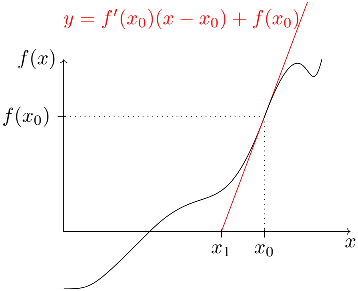
Newton's Method

To go from x_n to x_{n+1}, we write the equation of the tangent at the point $(x_n, f(x_n))$

$$y = f'(x_n)(x - x_n) + f(x_n),$$

x_{n+1} is such that $y = 0$, $\Rightarrow f'(x_n)(x_{n+1} - x_n) + f(x_n) = 0 \Rightarrow x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.



3. Limits and the Derivative

Learning Objectives

- Evaluate increments.

Learning Objectives

- Evaluate increments.
- Evaluate differentials.

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- Evaluate increments.
- Evaluate differentials.
- Use differentials to approximate increments.

EXAMPLE 1

Let $y = f(x) = x^2$.

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If x changes from 2 to 2.5, then y will change from $y = f(2) = 4$ to $y = f(2.5) = 6.5$.

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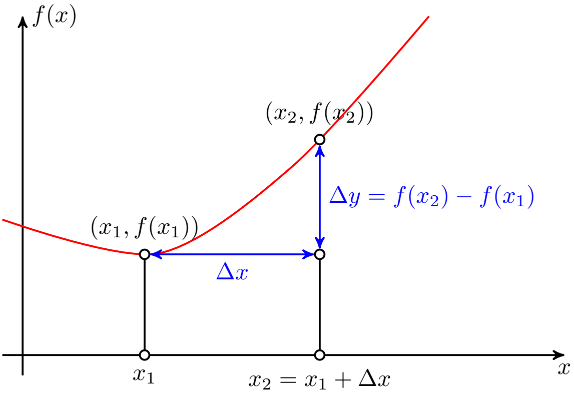
In this example,

$$\Delta x = 2.5 - 2 = 0.5$$
$$\Delta y = f(2.5) - f(2) = 6.5 - 4 = 1.5$$

DEFINITION **Increments**

For $y = f(x)$, $\Delta x = x_2 - x_1$, so $x_2 = x_1 + \Delta x$ and $\Delta y = y_2 - y_1 = f(x_2) - f(x_1)$

- Δy represents the change in y corresponding to a Δx change in x.
- Δx can be either positive or negative.



Differentials

Assume that the limit

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

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Multiplying both sides of this equation by Δx gives us

$$\Delta x \cdot f'(x) \approx \cancel{\Delta x} \frac{\Delta y}{\cancel{\Delta x}}$$

$$\Delta y \approx f'(x) \Delta x$$

Here the increments Δx and Δy represent the actual changes in x and y.

Differentials (continued)

One of the notation for the derivative is: $f'(x) = \frac{dy}{dx}$.
Multiplying both sides of this equation by dx gives us

$$dy = f'(x)dx$$

We treat this equation as a definition, and call dx and dy differentials.

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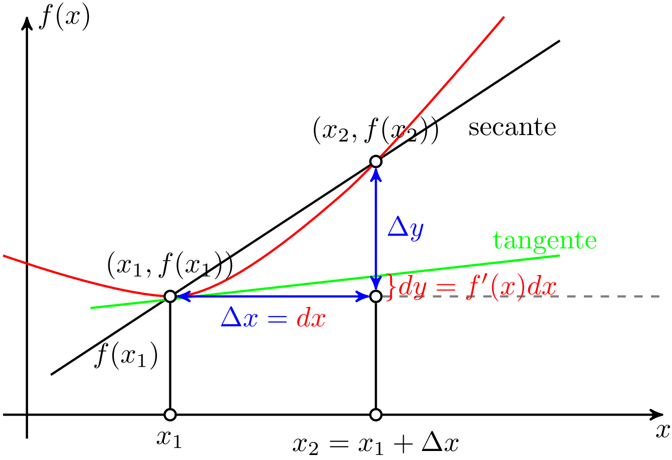
In application, we use dy (which is easy to calculate) to estimate Δy (which is what we want).



$f(x)$
$(x_2, f(x_2))$
secante
Δy
tangente
$(x_1, f(x_1))$
$\}dy = f'(x)dx$
$\Delta x = dx$
$f(x_1)$
x_1
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x

EXAMPLE 2

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Find dy for $f(x) = x^2 + 6x$ and evaluate dy for $x = 2$ and $dx = 0.1$.
Using the definition of the differential, we have

$$dy = f'(x)dx = (2x + 6)dx$$

When $x = 2$ and $dx = 0.1$, $dy = (2(2) + 6)\frac{1}{10} = 1$

EXAMPLE 3

A company manufactures and sells x laptops per week. If the weekly cost and revenue equations are

$$\begin{cases} C(x) = 5000 + 2x \\ R(x) = 10x - \frac{x^2}{1000} \\ 0 \leq x \leq 8000 \end{cases} \tag{1}$$

Find the approximate changes in revenue and profit if production is increased from 1000 to 1010 units/week.

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We'll approx ΔR and ΔP with dR and dP.

$$dR(x) = R'(x)dx = \frac{d}{dx}\left(10x - \frac{x^2}{1000}\right)dx = (10 - \frac{x}{500})dx$$

$$dP(x) = P'(x)dx = (8 - \frac{x}{500})dx$$

Here $x = 1000$ and $dx = 10$, $dR = (10-2)10 = \$80$/week $dP = (8-2)10 = \$60$/week.

EXERCISE

1. Find the indicated quantities for $f(x) = 2x^2$
   a. Δy, Δx and $\frac{\Delta y}{\Delta x}$; given $x_1 = 2$ and $x_2 = 6$.
   b. $\frac{f(x_1+\Delta x)-f(x_1)}{\Delta x}$; given $x_1 = 2$.
2. Evaluate dy and Δy for $f(x) = x^2 - 5$, where $x = 5$ and $dx = \Delta x = 0.01$.
3. Find Δy and dy for $y = x - x^2$ when $x = 1$.
4. The total monthly profit (in dollars) that Mandy's Painted Murals earns when the company is contracted to paint x murals in a month can be modelled by $P(x) = -5x^2 + 500x$, where $0 \leq x \leq 100$. Use dP to approximate the change in profit if production increased from 40 to 50 murals per month. Compare this value with the actual change in profit ΔP.

3. Limits and the Derivative

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- Solve applications involving marginal cost/revenue/profit.

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- Solve applications involving marginal average cost/revenue/profit.

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DEFINITION **Marginal Cost**

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$$\text{Total cost} = C(x)$$
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If x is the number of units of a product produced and sold in some time interval, then

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Marginal Cost & Exact Cost

Assume that $C(x)$ is the total cost of producing x items. Then the exact cost of producing the $(x+1)$ item is

$$C(x+1) - C(x).$$

The marginal cost is an approximation of the exact cost.

$$C'(x) \approx C(x+1) - C(x).$$

Note that similar statement are true for revenue and profit

EXAMPLE 1

The total cost of producing x guitar is

$$C(x) = 1000 + 100x - 0.25x^2$$

1. Find the exact cost of producing the 51^{st} guitar
2. Use the marginal cost to approximate the cost of producing the 51^{st} guitar

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1. The exact cost is

$$\begin{aligned} C(x+1) - C(x) &= 1000 + 100(x+1) - 0.25(x+1)^2 - 1000 - 100x + 0.25x^2 \\ &= 100 - 0.5x - 0.25 = 99.75 - 0.5x \end{aligned}$$

So $C(51) - C(50) = 99.75 - 0.5 \cdot 50 = \$74.75$

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   So $C(51) - C(50) = 99.75 - 0.5 \cdot 50 = \$74.75$

2. The marginal cost is $C'(x) = 100 - 0.5x$.
   So $C'(50) = 100 - 25 = \$75$

DEFINITION **Marginal Average Cost**

If x is the number of units of a product produced in some time interval, then

$$\text{Average cost per unit} = \overline{C}(x) = \frac{C(x)}{x}$$

$$\text{Marginal average cost} = \overline{C'}(x) = \frac{d}{dx}\overline{C}(x)$$

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$$\text{Average profit per unit} = \overline{P}(x) = \frac{P(x)}{x}$$

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WARNING!

To calculare the marginal average, you must calculate the average first and then the derivative.

EXERCISE

The total cost of printing x dictionaries is

$$C(x) = 20000 + 10x$$

1. Find the average cost per unit if 1000 dictionaries are produced.
2. Find the marginal average cost at a production level of 1000 dictionaries.
3. Use the results from above to estimate the average cost per dictionary if 1001 dictionaries are produced.

EXERCISE

The total cost of printing x dictionaries is

$$C(x) = 20000 + 10x$$

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1.

$$\overline{C}(x) = \frac{C(x)}{x} = \frac{20000 + 10x}{x}$$

$$\overline{C}(1000) = \frac{20000 + 10000}{1000} = \$30$$

EXERCISE

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2 The marginal average cost is

$$\overline{C'}(x) = \frac{d}{dx}\left(\frac{C(x)}{x}\right) = \frac{d}{dx}\left(\frac{20000 + 10x}{x}\right) = \frac{-2000}{x^2}$$
$$\overline{C'}(1000) = \frac{-20000}{1000^2} = -0.02$$

This mean that is you raise production from 1000 to 1001 dictionaries, the price per book will fall approximately 2 cents.

EXERCISE

The total cost of printing x dictionaries is

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EXERCISE

The total cost of printing x dictionaries is

$$C(x) = 20000 + 10x$$

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3 Average cost for 1000 dictionaries $= \$30$
Marginal average cost $= -0.02$
The average cost per dictionary for 1001 dictionaries would be the average for 1000, plus the marginal average cost, or

$$\$30 + \$(-0.02) = \$29.98$$

EXERCISE

The price-demand equation and the cost function for production of television sets are given by

$$p(x) = 300 - \frac{x}{30}, \quad \text{and} \quad C(x) = 150000 + 30x$$

where x is the number of sets that can be sold at a price $\$p$ per set, and $C(x)$ is the total cost of producing x sets.

1. Find marginal cost.
2. Find the revenue function in terms of x.
3. Find the marginal revenue.
4. Find $R'(1500)$.
5. Find the profit function in terms of x.
6. Find the marginal profit.
7. Find $P'(1500)$.