Mathématiques pour SHS

Master Sciences des données et histoire

Fabien Navarro

E-mail: fabien.navarro@univ-paris1.fr

UNIVERSITÉ PARIS 1 PANTHÉON SORBONNE

Notes

4. Additional Derivative Topics

Notes

4. Additional Derivative Topics

4-1 The Constant e and Continuous Compound Interest

Learning Objectives

- solve problems involving the irrational number e.
- solve problems involving continuous compound interest.

Notes

4. Additional Derivative Topics

4-1 The Constant e and Continuous Compound Interest

Simple Interest

Deposit amount P into bank account.
Interest is earned, but only on P (you don't earn interest on the interest).
The interest is kept separate from the principal P.
Let

- $r =$ interest rate expressed as a decimal.
- $t =$ time in years since the deposit;
- $A =$ amount of money in the account at time t.

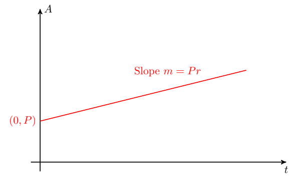
Simple Interest formula:

- $A =$ Principal + Interest
- $A = P + Prt$
- $A = P(1 + rt)$ (factored form)

Notes

Simple Interest

- $A = P(1 + rt)$ (factored form)
- $A = (Pr)\,t + P$ (rewritten in form $y = mx + b$)



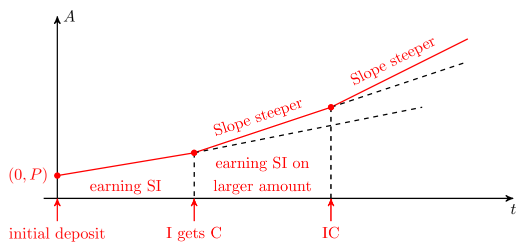
EXAMPLE 1

Deposit $\$1000$ into account with 2% SI. What will be the balance after 5 years?
$P = 1000$, $r = 0.02$, $t = 5$, $A =$unknown.
$A = 1000(1 + (0.02)(5)) = 1000(1 + 0.1) = 1000(1.1) = 1100\$$

Notes

Compound Interest (Periodically Compounded Interest)

Every once in awhile, the interest gets added to the bucket of money that is earning interest, so then interest will accumulate faster after that.



Notes

General Formula for Compound Interest

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

- $P =$ principal$=$initial deposit.
- $r =$ interest rate expressed as a decimal.
- $t =$ time in years since the deposit.
- $n =$ numbers of times per year that interest gets compounded.
- $A =$ amount of money in the account at time t.

EXAMPLE 2

Deposit $\$1000$ into account with 2% interest compounded annually. What will be the balance after 5 years?
$P = 1000$, $r = 0.02$, $t = 5$, $n = 1$, $A =$unknown.
$A = 1000\left(1 + \frac{0.02}{1}\right)^{1\cdot 5} = 1000(1 + 0.2)^5 \approx \$1104.08$

Notes

EXAMPLE 3

Same deposit, same interest, same time t but now compounded monthly.
$P = 1000$, $r = 0.02$, $t = 5$, $n = 12$, $A =$unknown.
$A = 1000\left(1 + \frac{0.02}{12}\right)^{12\cdot 5} \approx \$1105.08$

EXAMPLE 4

Same deposit, same interest, same time t but now compounded daily.
$P = 1000$, $r = 0.02$, $t = 5$, $n = 365$, $A =$unknown.
$A = 1000\left(1 + \frac{0.02}{365}\right)^{365\cdot 5} \approx \$1105.17$

REMARK

Notice the trend in the previous examples :
$P = 1000$, $r = 0.02$, $t = 5$
$\underset{\text{(SI, never compounded)}}{1100} < \underset{(n\,=\,1\text{ compounded yearly})}{1104.08} < \underset{(n\,=\,12\text{ compounded monthly})}{1105.08} < \underset{(n\,=\,365\text{ compounded daily})}{1105.17}$

Notes

Clearly as we compound more frequently, the balance after 5 years gets bigger.
But it seems like as n gets really big, the balance doesn't increase so much, and is may be ever levelling off.
That leads to a question:
What is

$$\lim_{n\to\infty} A?$$

That is, what is

$$\lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^{nt}?$$

Related question: what is

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{n}?$$

Investigate with a calculator

n	1	10	100	10000	...
$(1+\frac{1}{n})^n \approx$					...

Notes

Clearly as we compound more frequently, the balance after 5 years gets bigger.
But it seems like as n gets really big, the balance doesn't increase so much, and is may be ever levelling off.
That leads to a question:
What is

$$\lim_{n\to\infty} A?$$

That is, what is

$$\lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^{nt}?$$

Related question: what is

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{n}?$$

Investigate with a calculator

n	1	10	100	10000	...
$(1+\frac{1}{n})^n \approx$	2	2.59374	2.70481	2.71692	...

These values seem to be getting closer and closer to some number that is around 2.71

Notes

Huge Fact of math

The limit of $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ does exist.
It is given the symbol e

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{n} = e$$

Fact about e:

- it is a real number
- it is irrational (cannot be expressed as a ratio of integers or as a terminating or repeating decimal)
- $e \approx 2.718$

Related limits

$$\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^{n} = e^x$$

$$\lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^{nt} = Pe^{rt}$$

(this expr contains no n)

Introduce a new kind of bank account interest: Continuously Compounded Interest

Notes

EXAMPLE 5

Same deposit, same interest, same time t but now Continuous Compound Interest.
$P = 1000$, $r = 0.02$, $t = 5$, $A =$unknown.
$A = 1000e^{0.02\cdot 5} = 1000e^{0.1} \approx \$1105.17$

Notes

Notes

Examples involving Continuously Compounded Interest

$$A = Pe^{rt}$$

Equation involving A, P, r, t solved for A.

Solve for P

$$A = Pe^{rt} \quad \text{(divide by } e^{rt}\text{)}$$
$$P = \frac{A}{e^{rt}}$$

Equation involving A, P, r, t solved for P.

Notes

Solve for r

$$A = Pe^{rt} \quad \text{(divide by } P\text{)}$$
$$\frac{A}{P} = e^{rt} \quad \text{(take log of both sides)}$$
$$\ln\left(\frac{A}{P}\right) = \ln\left(e^{rt}\right) \quad (\ln(e^x) = x)$$
$$\ln\left(\frac{A}{P}\right) = rt \quad \text{(divide by } t\text{)}$$
$$r = \frac{\ln\left(\frac{A}{P}\right)}{t}$$

Notes

Solve for t (steps like on previous example)

$$\ln\left(\frac{A}{P}\right) = rt \quad \text{(divide by } t\text{)}$$
$$t = \frac{\ln\left(\frac{A}{P}\right)}{r}$$

Notes

EXAMPLE 6

Deposit $\$937$ into account with 2.3% interest compounded continuously.
Find the balance after 7 years.
We have $P = 937, r = 0.023, t = 7$ and $A =$unknown, so use form $A = Pe^{rt}$.

$$A = 937e^{(0.023)7} \approx \$1100.68$$

EXAMPLE 7

Deposit $\$937$ into account with 2.3% interest compounded continuously.
How long until the balance is $\$1200$?
We have $P = 937, r = 0.023, t =$unknown and $A = 1200$, so use form $t = \frac{\ln\left(\frac{A}{P}\right)}{r}$.

$$t = \frac{\ln\left(\frac{1200}{987}\right)}{0.023} \approx 10.75 \text{ years}$$

EXAMPLE 8

Deposit some money into account with 2.3% interest compounded continuously. How long until the balance double?
We have $P =$unknown, $r = 0.023, t =$unknown and $A = 2P$unknown, so use form $t = \frac{\ln\left(\frac{A}{P}\right)}{r}$.

$$t = \frac{\ln\left(\frac{2P}{P}\right)}{0.023} = \frac{\ln(2)}{0.023} \approx 30.14 \text{ years}$$

EXAMPLE 9

If you want an account with continuously compounded interest to double in 20 years, what interest rate will you need?
We have $P =$unknown, $r =$unknown, $t = 20$ and $A = 2P$, so use form $r = \frac{\ln\left(\frac{A}{P}\right)}{t}$.

$$r = \frac{\ln\left(\frac{2P}{P}\right)}{20} = \frac{\ln(2)}{20} \approx 0.0347$$

Notes

EXERCISES

1. Solve for t or r to two decimal places. a) $4 = e^{0.56t}$, b)$3 = e^{15t}$, c)$5 = e^{0.0456t}$
2. Carolina Bank offers a $10-$years Certificates of Deposit (CD) that earns 7% compounded continuously. If $\$2500$ is invested in this CD, how much will it be worth in 10 years?
3. A saving bond will pay $\$10,000$ maturity 20 years from now. How much should you be willing to pay for the note now if money is worth 3.85% compounded semiannually?
4. A couple paid $\$30,000$ for a house forty years ago. Today, they sold the house for $\$250,000$. If interest is compounded continuously, what annual nominal rate of interest did the original $\$30,000$ investment earn?
5. At what nominal interest rate compounded continuously must Jennifer invest her savings so that her money doubles in 7 years?
6. Jocelyn invests $\$1000$ in an account that earns 3.75% compounded monthly and $\$2500$ in an account that earns 3.9% compounded continuously. Use graphical approximation methods to determine how long it will take for her total investment to grow to $\$5000$?

Notes

Learning Objectives

- Find the derivative of e^x
- Find the derivative of $\ln(x)$
- Find the derivatives of other logarithmic and exponential functions.

Notes

Exponential function: Rule #1

Two eq version: If $f(x) = e^x$, then $f'(x) = e^x$

Sing eq version: $\frac{d}{dx}e^x = e^x$

Exponential functions with bases other than e may also be differentiated.

Exponential function: Rule #2

Two eq version: If $f(x) = b^x$, then $f'(x) = b^x \ln(b)$

Sing eq version: $\frac{d}{dx}b^x = b^x \ln(b)$

Notes

EXAMPLE 1

Find the derivative of the following functions.

1. $f_1(x) = e^x/2$
2. $f_2(x) = e^{x/2}$
3. $f_3(x) = 2e^x + x^2$
4. $f_4(x) = -7x^e - 2e^x + e^2$

$$f_1'(x) = e^x/2$$
$$f_2'(x) = (1/2)e^{x/2}$$
$$f_3'(x) = 2e^x + 2x$$
$$f_4'(x) = -7ex^{e-1} - 2e^x$$

Remember that e is a real number, so the power rule is used to find the derivative of x^e. The derivative of the exponential function e^x, on the other hand, is e^x. Note also that e^2 is a constant, so its derivative is 0.

Notes

EXERCICES

1. Find the derivative of the following functions.
   a) $y = 11e^x$
   b) $f(x) = 11x^e$
   c) $f(x) = 11e^{12}$
   d) $f(x) = 11 \cdot 12^x$
   e) $f(x) = 11e^{12x}$
2. The value of a truck is given by the formula

$$S(t) = 174,000(0.9)^t$$

   where $t =$time in years since truck was purchased.
   a) What was the purchase price of the truck?
   b) What is the value of the truck after 5 years?
   c) What is the rate of change of the value of the truck at time $t = 5$years?

Notes

Logarithm function

We summarize important facts about logarithmic functions:

Recall that the inverse of an exponential function is called a logarithmic function. For $b > 0$ and $b \neq 1$

Logarithmic form is equivalent to **Exponential form**

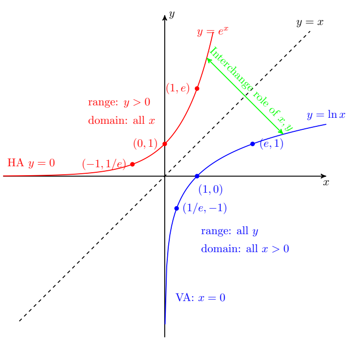
$y = \log_b(x)$	$x = b^y$
Domain$(0, \infty)$	Domain$(-\infty, \infty)$
Range$(-\infty, \infty)$	Range$(0, \infty)$

The base we will be using is e.

$$\ln(x) = \log_e(x)$$

Notes

Logarithm & Exponential



Notes

Logarithm function: Rule #1

Two eq version: If $f(x) = \ln(x)$, then $f'(x) = \frac{1}{x}$

Sing eq version: $\frac{d}{dx}\ln(x) = \frac{1}{x}$

Logarithmic functions with bases other than e may also be differentiated.

Logarithm function: Rule #2

Two eq version: If $f(x) = \log_b(x)$, then $f'(x) = \frac{1}{x\ln(b)}$

Sing eq version: $\frac{d}{dx}\log_b(x) = \frac{1}{x\ln(b)}$

Notes

EXAMPLE 2

Find the derivative of the following functions.

1. $f_1(x) = 5\ln(x)$
2. $f_2(x) = x^2 + 3\ln(x)$
3. $f_3(x) = 10 - \ln(x)$
4. $f_4(x) = x^4 - \ln x^4$

$$f_1'(x) = \frac{5}{x}$$
$$f_2'(x) = 2x + \frac{3}{x}$$
$$f_3'(x) = \frac{-1}{x}$$
$$f_4'(x) = 4x^3 - \frac{4}{x}$$

Before taking the last derivative, we rewrite $f(x)$ using a property of logarithms: $\ln x^4 = 4\ln x$

Notes

EXERCISES

Find the derivative of the following functions.

1. $y = 12\ln(x)$
2. $f(x) = 12\log_{13}(x)$
3. $f(x) = 12\log(x)$
4. $f(x) = 12\ln(13)$
5. $f(x) = 12\ln(13x)$
6. $y = \ln(x^{10}) + \ln(x^3)$
7. $f(x) = \ln(2x^{15})$

Notes

EXERCISES

1. Simplify and calculate
$$\left(\frac{1}{49}\right)^{\frac{-3}{2}}, \left(\frac{1}{3}\right)^{-0.8}, \left(\frac{32}{243}\right)^{\frac{-2}{5}}, (0.64)^{-0.5}, \left(\frac{27}{64}\right)^{\frac{2}{3}}, \sqrt[7]{0.078125}, \frac{\sqrt[3]{2}\sqrt[5]{64^3}\sqrt[4]{8^5}}{\sqrt[5]{16}\sqrt[3]{16^4}\sqrt[20]{2048}}$$
2. Draw graphs of the following functions:
$f_1(x) = x^2, f_2(x) = x^3, f_3(x) = x^{-1}, f_4(x) = x^{-2}$
3. Write the equations of tangents to curves representing the functions at the points $x = 1$ and $x = -2$. Plot these tangents.
4. Simplify the following expressions:
$$\frac{1}{\ln(10)}\ln(10^x), \log(2) + \ln(0.5), \ln[\ln(10^{e^x})], \ln(10)\log_{10}(e^{10}).$$

Notes

1.

$$\left(\frac{1}{49}\right)^{\frac{-3}{2}} = 49^{\frac{3}{2}} = (7^{\cancel{2}})^{\frac{3}{\cancel{2}}} = 7^3 = 343$$

$$\left(\frac{1}{32}\right)^{-0.8} = \left(\frac{1}{32}\right)^{\frac{-4}{5}} = 32^{4/5} = (2^5)^{4/5} = 2^4 = 16$$

$$\left(\frac{32}{243}\right)^{\frac{-2}{5}} = \left(\frac{2^5}{3^5}\right)^{\frac{-2}{5}} = \left(\frac{2}{3}\right)^{-2} = \left(\frac{3}{2}\right)^{2} = \frac{9}{4} = 2.25$$

Notes

$$(0.64)^{-0.5} = \left(\frac{64}{100}\right)^{-1/2} = \left(\frac{8^2}{10^2}\right)^{-1/2} = \left(\frac{8}{10}\right)^{-1} = \frac{5}{4} = 1.25$$

$$\left(\frac{27}{64}\right)^{2/3} = \left(\frac{3^3}{4^3}\right)^{2/3} = \left(\frac{3}{4}\right)^{2} = \frac{9}{16} = 0.5625$$

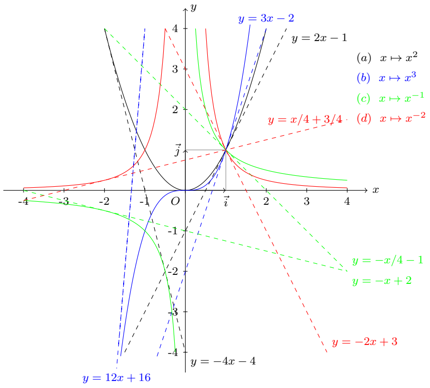
$$\sqrt[7]{0.078125} = 0.078125^{1/7} = \left(\frac{5^7}{10^7}\right)^{1/7} = \frac{5}{10} = \frac{1}{2}$$

Notes

$$\begin{aligned}
\frac{\sqrt[3]{2}\,\sqrt[5]{64^3}\,\sqrt[4]{8^5}}{\sqrt[5]{16}\,\sqrt[3]{16^4}\,\sqrt[20]{2048}} &= \frac{2^{1/3}64^{3/5}8^{5/4}}{16^{1/5}16^{4/3}2048^{1/20}} \\
&= \frac{2^{1/3}2^{18/5}2^{15/4}}{2^{4/5}2^{16/3}2^{11/20}} \\
&= 2^{1/3-16/3}2^{18/5-4/5}2^{75/20-11/20} \\
&= 2^{-5}2^{14/5}2^{16/5} \\
&= 2^{-5}2^{6} \\
&= 2
\end{aligned}$$

Notes

2.



Notes

4. Additional Derivative Topics
4-2 Derivatives of Exponential and Logarithmic Functions

3.
$y = 2x - 1$, $y = 3x - 2$, $y = -x + 2$, $y = -2x + 3$.
$y = -4x - 4$, $y = 12x + 16$, $y = -x/4 - 1$, $y = x/4 + 3/4$.

4.

$$\frac{1}{\ln(10)} \ln(10^x) = \frac{1}{\ln(10)} x \ln(10) = x$$

$$\ln(2) + \ln(0.5) = \ln(2) + \ln(\frac{1}{2}) = \cancel{\ln(2)} + \ln(1) - \cancel{\ln(2)} = 0$$

$$\ln[\ln(10^{e^x})] = \ln[e^x \ln(10)] = \ln(e^x) + \ln[\ln(10)] = x + \ln[\ln(10)]$$

$$\ln(10)\log_{10}(e^{10}) = \cancel{\ln(10)} \frac{\ln(e^{10})}{\cancel{\ln(10)}} = 10\ln(e) = 10$$

Notes

4. Additional Derivative Topics
4-3 Derivatives of Products and Quotients

Learning Objectives
- Find the derivative of the product of two functions
- Find the derivative of the quotient of two functions

Notes

4. Additional Derivative Topics
4-3 Derivatives of Products and Quotients

THEOREM 1: The Product Rule

Goal: take the derivative of a product of functions $f(x) \cdot g(x)$
The obvious way:

$$\cancel{\frac{d}{dx}(f(x) \cdot g(x)) = \left(\frac{d}{dx} f(x)\right)\left(\frac{d}{dx} g(x)\right)}$$

is wrong!!
The Product Rule

$$\frac{d}{dx}(f(x) \cdot g(x)) = \left(\frac{d}{dx} f(x)\right) g(x) + f(x)\left(\frac{d}{dx} g(x)\right) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

In words: The derivative of the product of two functions is the first function times the derivative of the second function plus the second function times the derivative of the first function.

Notes

4. Additional Derivative Topics
4-3 Derivatives of Products and Quotients

EXAMPLE 1

Find the derivative of

$$f(x) = (-3x^2 + 5x - 7) \cdot (3x - 2)$$

using the product rule.

$$\begin{aligned} f'(x) &= \left(\frac{d}{dx}(-3x^2 + 5x - 7)\right) \cdot (3x - 2) + (-3x^2 + 5x - 7) \cdot \left(\frac{d}{dx}(3x - 2)\right) \\ &= (-6x + 5) \cdot (3x - 2) + (-3x^2 + 5x - 7) \cdot (3) \\ &= -18x^2 + 12x + 15x - 10 - 9x^2 + 15x - 21 \\ &= -27x^2 + 42x - 31 \end{aligned}$$

Notes

EXAMPLE 2

Find the derivative of

$$f(x) = (-3x^2 + 5x - 7) \cdot e^x$$

using the product rule.

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left((-3x^2 + 5x - 7) \cdot e^x\right) \\ &= \left(\frac{d}{dx}(-3x^2 + 5x - 7)\right) \cdot e^x + (-3x^2 + 5x - 7)\frac{d}{dx}e^x \\ &= (-6x + 5)e^x + (-3x^2 + 5x - 7)e^x \\ &= \left((-6x + 5) + (-3x^2 + 5x - 7)\right)e^x \\ &= (-3x^2 - x - 2)e^x \end{aligned}$$

Notes

EXAMPLE 3

1. Find the derivative of $f(x) = 5x^2 \ln(x)$
2. Find $f'(1)$ and $f'(e)$

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(5x^2 \ln(x)\right) \\ &= \left(\frac{d}{dx}5x^2\right)\ln(x) + (5x^2)\left(\frac{d}{dx}\ln(x)\right) \\ &= (10x)\ln(x) + (5x^2)\frac{1}{x} \\ &= 10x\ln(x) + 5x \\ &= 5x(2\ln(x) + 1) \end{aligned}$$

$$\begin{aligned} f'(1) &= 5 \cdot 1(2\ln(1) + 1) = 5 \\ f'(e) &= 5 \cdot e(2\ln(e) + 1) = 15e \end{aligned}$$

Notes

THEOREM 2: The Quotient Rule

Goal: take the derivative of a quotient of functions $\frac{\text{Top}(x)}{\text{Bot}(x)}$

The obvious way:

$$\cancel{\frac{d}{dx}\frac{\text{Top}(x)}{\text{Bot}(x)} = \frac{\frac{d}{dx}\text{Top}(x)}{\frac{d}{dx}\text{Bot}(x)}}$$

is wrong!!

The Quotient Rule

$$\frac{d}{dx}\frac{\text{Top}(x)}{\text{Bot}(x)} = \frac{\left(\frac{d}{dx}\text{Top}(x)\right)\text{Bot}(x) - \text{Top}(x)\left(\frac{d}{dx}\text{Bot}(x)\right)}{(\text{Bot}(x))^2}$$

In words: The derivative of the quotient of two functions is the bottom function times the derivative of the top function minus the top function times the derivative of the bottom function, all over the bottom function squared

Notes

EXAMPLE 4

Find the derivative of

$$f(x) = \frac{5x^2 - 1}{2x - 3}$$

using the Quotient Rule.

$$\begin{aligned} f'(x) &= \left(\frac{5x^2 - 1}{2x - 3}\right)' \\ &= \frac{(5x^2 - 1)'(2x - 3) - (5x^2 - 1)(2x - 3)'}{(2x - 3)^2} \\ &= \frac{(10x)(2x - 3) - (5x^2 - 1)(2)}{(2x - 3)^2} \\ &= \frac{20x^2 - 30x - 10x^2 - 2}{(2x - 3)^2} \\ &= \frac{10x^2 - 30x - 2}{(2x - 3)^2} \end{aligned}$$

Notes

EXERCISES

1. Let $f(x) = 7(x^2 + 3x + 5)$
   a) Find $f'(x)$, using the Product Rule to deal with the 7 in front.
   b) Start over. Find $f'(x)$ again, this time using the Constant Multiple Rule.
2. Let $f(x) = \frac{x^2+3x+5}{7}$
   a) Find $f'(x)$, using the Quotient Rule.
   b) Start over. Find $f'(x)$ again, this time using the Constant Multiple Rule.
3. Find the derivative of $f(x) = \frac{e^x}{3x^2-5}$.
4. Find the derivative of $f(x) = \frac{2x}{3^x}$.
5. Suppose a factory worker can assemble $I(t) = \frac{90t^2}{t^2+2}$ items after t days of training.
   a) Find $I(5)$ and $I'(5)$ and interpret the meaning of these values in the context of the problem.
   b) Use the results from part a to estimate the number of items the worker will be able to assemble after 7 days of training.

Notes

EXERCISES

1. Use the product rule or quotient rule to find $f'(x)$
   a) $f(x) = x^4(3x-1)$
   b) $f(x) = e^x \ln(x)$
   c) $f(x) = \frac{x^2}{7x+8}$
2. Find the values of x where $f'(x) = 0$ for $f(x) = \frac{4-2x}{x^2+1}$
3. Find the indicated derivative and simply
   a) $\frac{d}{dx} 3^x x^3$
   b) $\frac{d}{dx} \frac{x \ln x}{x+1}$
4. Find the equation of tangent line to $g(x) = x \ln x$ at $x = 1$
5. Let
$$f(x) = \frac{x}{x^2+25}$$
   a) Find $f'(x)$ and $f'(0)$.
   b) Find the $x-$coordinate of all points on the graph of f that have horizontal tangent lines.

Notes

3.
$$\frac{d}{dx} 3^x x^3 = 3^x x^2(3 + x\ln(x))$$
$$\frac{d}{dx} \frac{x \ln x}{x+1} = \frac{\ln x - x + 1}{(x+1)^2}$$

4. $y = x - 1$

5.
   a) $f'(x) = \frac{-(x^2-25)}{(x^2+25)^2}$, $f'(0) = \frac{1}{25}$, this tells us that at $x = 0$, the tangent line has slope $m = \frac{1}{25}$.
   b) Horizontal lines have known slope $m = 0$. But we know that $m = f'(a)$ where "a" is the $x-$coord of the point of tangency. So we want to know all values of "a" such that $f'(a) = 0$. We can just set $f'(x) = 0$ and solve for x.
$$\frac{-(x^2-25)}{(x^2+25)^2} = 0$$
   Consider the denominator first: $x^2 \geq 0$ always!
   so $x^2 + 25 \geq 25$ always.
   so denominator $(x^2+25)^2 \geq 25^2$ always.
   So the denominator will never be zero.

Notes

Now consider the numerator:
we need numerator$= 0$
$$-(x^2-25) = 0$$
$$x^2 - 25 = 0$$
$$(x+5)(x-5) = 0$$
This equation has two solutions: $x = 5$ and $x = -5$.
Conclusion: The graph of f has horizontal tangent lines at $x = -5$ and $x = 5$.

Notes

Notes

Learning Objectives

- Identify and define composite functions.
- Use the general power rule when appropriate to find the derivative of functions.
- Use the chain rule to find the derivative of composite functions.

Notes

DEFINITION: **Composite Functions**

A function m is a composite of functions f and g if

$$m(x) = (f \circ g)(x) = f[g(x)]$$

The domain of m is the set of all numbers x such that x is in the domain of g and $g(x)$ is in the domain of f.

EXAMPLE 1

$f(x) = x + \frac{1}{x}$, $g(x) = \frac{x+1}{x+2}$

$$\begin{aligned} f[g(x)] &= \frac{x+1}{x+2} + \frac{1}{\frac{x+1}{x+2}} \\ &= \frac{x+1}{x+2} + \frac{x+2}{x+1} \\ &= \frac{(x+1)(x+1)+(x+2)(x+2)}{(x+1)(x+2)} \\ &= \frac{2x^2+6x+5}{(x+1)(x+2)} \end{aligned}$$

Notes

We have already made extensive use of the power rule:

$$\frac{d}{dx}x^n = nx^{n-1}$$

Now we want to generalize this rule so that we can differentiate composite functions of the form $[u(x)]^n$.
Is the power rule still valid if we replace x by a function $u(x)$?

EXAMPLE 2

Let $u(x) = 3x^3$ and $f(x) = [u(x)]^3 = \left(3x^3\right)^3 = 27x^9$. Which of the following is $f'(x)$?

$$\text{(a) } 3[u(x)]^2, \quad \text{(b) } 3[u'(x)]^2, \quad \text{(c) } 3[u(x)]^2u'(x)$$

We know that $f'(x) = 243x^8$

(a) $3[u(x)]^2 = 3(3x^3)^2 = 27x^6$.This is not correct
(b) $3[u'(x)]^2 = 3(9x^2)^2 = 243x^4$.This is not correct
(c) $3[u(x)]^2u'(x) = 27x^6 9x^2 = 243x^8$.This is correct

Notes

THEOREM 1: Generalized Power Rule

What we have seen is an example of the generalized power rule: if u is a function of x, then

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}$$

For example, $\frac{d}{dx}(x^2+3x+5)^3 = 3(x^2+3x+5)^2(2x+3)$
Here u is x^2+3x+5 and $\frac{du}{dx} = 2x+3$.

We have used the generalized power rule to find derivatives of composite functions of the form $f(g(x))$ where $f(u) = u^n$ is a power function. But what if f is not a power function?
It is a more general rule, the chain rule, that enables us to compute the derivatives of many composite functions of the form $f(g(x))$.

THEOREM 2: The Chain Rule

Goal: Find the derivative of a nested function.
That is a function of the form

$$f[g(x)] \quad (\text{outer}(\text{inner}(x)))$$

The Chain Rule
If $y = f(u)$ and $u = g(x)$ define the composite function $y = m(x) = f(u) = f[g(x)]$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}, \text{ provided } \frac{dy}{du} \text{ and } \frac{du}{dx} \text{ exist}$$

$$\left(\frac{d}{dx}\text{outer}(\text{inner}(x)) = \text{outer}'(\text{inner}(x)) \cdot \text{inner}'(x)\right)$$

$$(m'(x) = f'[g(x)]g'(x))$$

Notes

General Derivative Rules

$$\frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1}f'(x)$$

$$\frac{d}{dx}\ln[f(x)] = \frac{1}{f(x)}f'(x)$$

$$\frac{d}{dx}e^{f(x)} = e^{f(x)}f'(x)$$

Notes

EXERCISES

1. Find the derivatives of the following functions:
   a) $h(x) = 2(3x^4 + 5x^2 + 6)^7$
   b) $h(x) = \frac{2}{(3x^4+5x^2+6)^7}$
   c) $h(x) = 2\sqrt{x^2 - 3x + 2}$
   d) $h(x) = e^{kx}$
   e) $h(x) = 7\ln(5x^2 - 30x + 65)$
   f) $h(x) = x^3(x^5 + 25)^7$
   h) $h(x) = \frac{x^3+7}{x^3}$
2. Let $f(x) = e^{-x^2+4x-4}$
   a) Find $f'(x)$.
   b) Find slope of line tangent to graph of f at $x = 0$.
   c) Find x-coordinates of all points of f that have horizontal tangent lines.

Notes

EXERCISES

1. Write each composite function in the form $y = f(u)$ and $u = g(x)$
   a) $y = (x^3 + 2x^2)^5$
   b) $y = e^{9x+8}$
   c) $y = \ln(19 - 7x)$
2. Use the Chain Rule to find $\frac{dy}{dx}$ for the composite functions in 1.
3. Find the derivative of the following functions.
   a) $h(x) = \sqrt[3]{3x - 1}$
   b) $h(x) = \ln(2x^4 - 7x^3 + 1)$
   c) $h(x) = e + e^{2x^2-1}$
4. Find the indicated derivative and simplify.
   a) $\frac{d}{dx}\frac{\ln(2x)}{x^2}$
   b) $\frac{d}{dw}\frac{w}{\sqrt{4w+2}}$
5. The amount of money in Lori's savings account after t years can be modeled by the equation $A = 10,000(1.008)^{4t}$. Find $A(5)$ and $A'(5)$ and interpret the results in the context of the problem

Notes

4. Additional Derivative Topics

4-4 The Chain Rule

1.
   a) $y = u^5, u = x + 2x^2 5$
   b) $y = e^u, u = 9x + 8$
   c) $y = \ln(u), u = 19 - 7x$
2.
   a) $\frac{dy}{dx} = 5u^4 u' = 5(x^3 + 2x^2)^4(3x^2 + 4x)$
   b) $\frac{dy}{dx} = u' e^u = 9e^{9x+8}$
   c) $\frac{dy}{dx} = \frac{u'}{u} = \frac{-7}{19-7x}$
3.
   a) $h(x) = \sqrt[3]{3x-1} = (3x-1)^{1/3}, h'(x) = \frac{1}{3}(3x-1)^{-2/3}3 = \frac{1}{(3x-1)^{2/3}}$
   b) $h'(x) = \frac{8x-21x^2}{2x^4-7x^3+1}$
   c) $h'(x) = 4xe^{2x^2-1}$
4.
   a) $\frac{d}{dx}\frac{\ln(2x)}{x^2} = \frac{\frac{1}{x}x^2 - \ln(2x)2x}{x^4} = \frac{x - 2x\ln(2x)}{x^4} = \frac{1-2\ln(2x)}{x^3}$

Notes

4. Additional Derivative Topics

4-4 The Chain Rule

b)

$$
\begin{aligned}
\frac{d}{dw}\frac{w}{\sqrt{4w+2}} &= \frac{1(4w+2)^{1/2} - w \cdot \frac{1}{2} \cdot 4(4w+2)^{-1/2}}{(4w+2)} \\
&= (4w+2)^{-1/2} - 2w(4w+2)^{-3/2} \\
&= \frac{(4w+2) - 2w}{(4w+2)^{3/2}} \\
&= \frac{2w+2}{(4w+2)^{3/2}}
\end{aligned}
$$

Notes

4. Additional Derivative Topics

4-5 Implicit Differentiation

Learning Objectives

- Use implicit differentiation to determine derivatives.

Notes

4. Additional Derivative Topics

4-5 Implicit Differentiation

Function Review and New Notation

So far, the equation of a curve has been specified in the form

$$y = x^2 - 5x, \quad \text{or} \quad f(x) = x^2 - 5x$$

This is called the explicit form. y is given as a function of x.
However, graphs can also be specified by equations of the form $F(x, y) = 0$

$$F(x, y) = x^2 + y^2 - 1 = 0$$

This is called the implicit form. You may or may not be able to solve for y.

Notes

Explicit and Implicit Differentiation

Consider the equation
$$y = x^2 - 5x$$
We can use the derivative to find y'
$$y' = 2x - 5$$
This is called explicit differentiation.
We can also rewrite the original equation as
$$F(x, y) = x^2 - 5x - y = 0$$
one can totally differentiate this equation.
This is called implicit differentiation.

Notes

EXAMPLE 1

Consider the equation
$$x^2 - 5x - y = 0$$
We will now differentiate both sides of the equation with respect to x, and keep in mind that y is supposed to be a function of x.
$$\begin{aligned} \frac{d}{dx}\left[x^2 - 5x - y\right] &= \frac{d}{dx}0 \\ 2x - \frac{dy}{dx} - 5 &= 0 \\ \frac{dy}{dx} &= 2x - 5 \end{aligned}$$
This is the same answer we got by explicit differentiation on the previous slide.

Notes

EXAMPLE 2

Consider the equation
$$x^2 - 3xy + 4y = 0$$
and differentiate implicitly.
$$\begin{aligned} \frac{d}{dx}\left[x^2 - 3xy + 4y\right] &= \frac{d}{dx}0 \\ \frac{d}{dx}x^2 - \frac{d}{dx}3xy + \frac{d}{dx}4y &= 0 \\ 2x - 3xy' - 3y + 4y' &= 0 \end{aligned}$$
Notice we used the product rule for the xy term.
Solve for y'
$$\begin{aligned} (3x - 4)y' &= 2x - 3y \\ y' &= \frac{2x - 3y}{3x - 4} \end{aligned}$$

Notes

EXAMPLE 3

Consider the equation
$$x^2 - 3xy + 4y = 0$$
and find the equation of the tangent at $(1, -1)$.

1. Confirm that $(1, -1)$ is a point on the graph.
2. Use the derivative from ex2 to find the slope of the tangent.
3. Use the point slope formula for the tangent.

1. $1^2 - 3(1)(-1) + 4(-1) = 1 + 3 - 4 = 0.$
2. $m = \frac{2\cdot 1 - 3\cdot(-1)}{3\cdot 1 - 4} = \frac{5}{-1} = -5.$
3. $$\begin{aligned} y - (-1) &= -5(x - 1) \\ y &= -5x + 4 \end{aligned}$$

Notes

EXAMPLE 4

Consider the equation $xe^x + \ln y + 3y = 0$ and differentiate implicitly.

$$\frac{d}{dx}\left[xe^x + \ln y + 3y\right] = \frac{d}{dx}0$$
$$\frac{d}{dx}xe^x + \frac{d}{dx}\ln y + \frac{d}{dx}3y = 0$$
$$xe^x + e^x + \frac{y'}{y} + 3y' = 0$$

Notice we used both the product rule (for the xe^x term) and the chain rule (for the $\ln y$).
Solve for y':

$$\frac{y'}{y} + 3y' = -e^x(x+1)$$
$$y'\left(\frac{1}{y} + 3\right) = -e^x(x+1)$$
$$y' = \frac{-e^x(x+1)}{\frac{1}{y} + 3}$$

Notes

NOTES

Why are we interested in implicit differentiation?

Why don't we just solve for y in terms of x and differentiate directly?

The answer is that there are many equations of the form $F(x, y) = 0$ that are either difficult or impossible to solve for y explicitly in terms of x, so to find y' under these conditions, we differentiate implicitly.

Notes

EXAMPLE 5

An example of an implicit function, for which implicit differentiation might be easier than attempting to use explicit differentiation, is

$$x^4 + 2y^2 = 8$$

In order to differentiate this explicitly with respect to x, one would have to obtain (via algebra)

$$y = f(x) = \pm\sqrt{\frac{8 - x^4}{2}}$$

and then differentiate this function. This creates two derivatives: one for $y > 0$ and another for $y < 0$.
One might find it substantially easier to implicitly differentiate the original function:

$$4x^3 + 4y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{-4x^3}{4y} = \frac{-x^3}{y}$$

Notes

EXAMPLE 6

Sometimes standard explicit differentiation cannot be used and, in order to obtain the derivative, implicit differentiation must be employed. An example of such a case is the equation:

$$y^5 - y = x$$

It is impossible to express y explicitly as a function of x and therefore $\frac{dy}{dx}$ cannot be found by explicit differentiation.
Using the implicit method, $\frac{dy}{dx}$ can be expressed:

$$5y^4\frac{dy}{dx} - \frac{dy}{dx} = \frac{dx}{dx}$$
$$\frac{dy}{dx}(5y^4 - 1) = 1$$
$$\frac{dy}{dx} = \frac{1}{5y^4 - 1}$$

Notes

EXERCISES

1. Differentiate the following equations implicitly and then solve for y'. To check your answer, solve the following equations for y and then differentiate directly.
   a. $2x + 3y - 4 = 0$
   b. $3x^3 - y = 10$
   c. $5x^5 + 2y - 1 = 0$
2. Use implicit differentiation to find y' and evaluate y' at the indicated point.
   a. $2x - xy = 3; \quad (1, -1)$
   b. $x^3y - y^3 + 8 = 0; \quad (0, 2)$
   c. $ye^x - x + 4 = 3y; \quad (0, 2)$
3. Find the equation of the tangent line to the graphs of the indicated equations at the given point.
   a. $x^2 - 4xy - 3 = 0; \quad (1, -\frac{1}{2})$
   b. $x^2y + x - 10 = 0; \quad (2, 2)$
4. Find the slope of the lines(s) tangent to the graph of $(y - x)^3 - y = x^2$ at $x = 0$.

Notes

EXERCISES

1. Use implicit differentiation to find y'
   a. $x^3 + y^3 = 4$
   b. $(x - y)^2 = x + y - 1$
   c. $y = x^2y^3 + x^3y^2$
   d. $e^{xy} = e^{4x} - e^{5y}$
   e. $x = \sqrt{x^2 + y^2}$
   f. $\frac{x-y^3}{y+x^2} = x + 2$
2. Find an equation of the line tangent to the graph of $(x^2 + y^2)^3 = 8x^2y^2$ at the point $(-1, 1)$.
3. Find an equation of the line tangent to the graph of $x^2 + (y - x)^3 = 9$ at $x = 1$

Notes

c.

$$\begin{aligned}
y' &= 2xy^3 + x^2 3y^4 y' + 3x^2y^2 + x^3 2yy' \\
y' - 3x^2y^2y' - 2x^3yy' &= 2xy^3 + 3x^2y^2 \\
y' &= \frac{xy^2(2y + 3x)}{1 - x^2y(3y + 2x)}
\end{aligned}$$

d.

$$\begin{aligned}
\frac{d}{dx}(xy)e^{xy} &= 4e^{4x} - \frac{d}{dx}(5y)e^{5y} \\
ye^{xy} + xy'e^{xy} &= 4e^{4x} - 5y'e^{5y} \\
y'(xe^{xy} + 5e^{5y}) &= 4e^{4x} - ye^{xy} \\
y' &= \frac{4e^{4x} - ye^{xy}}{xe^{xy} + 5e^{5y}}
\end{aligned}$$

Notes

e.

$$\begin{aligned}
1 &= \frac{1}{2}(x^2 + y^2)^{-1/2}(2x + 2yy') \\
1 &= \frac{x + yy'}{\sqrt{x^2 + y^2}} \\
\sqrt{x^2 + y^2} &= x + yy' \\
y' &= \frac{\sqrt{x^2 + y^2} - x}{y}
\end{aligned}$$

Notes

f. Clear the fraction by multiplying both sides of the equation by $y+x^2$, getting

$$\begin{aligned} x - y^3 &= (x+2)(y+x^2) \\ &= xy + x^3 + 2y + 2x^2 \end{aligned}$$

Now differentiate both sides of the equation

$$\begin{aligned} 1 - 3y^2y' &= y + xy' + 3x^2 + 2y' + 4x \\ y'(2+3y^2+x) &= 1 - y - 3x^2 - 4x \\ y' &= \frac{1-y-3x^2-4x}{2+3y^2+x} \end{aligned}$$

Notes

2. Differentiate both sides of the equation, getting

$$\begin{aligned} \frac{d}{dx}\left((x^2+y^2)^3\right) &= \frac{d}{dx}(8x^2y^2) \\ 3(x^2+y^2)^2(2x+2yy') &= 16xy^2 + 16x^2yy' \\ 6yy'(x^2+y^2)^2 - 16y'x^2y &= 16xy^2 - 6x(x^2+y^2)^2 \\ y' &= \frac{16xy^2 - 6x(x^2+y^2)^2}{6y(x^2+y^2)^2 - 16x^2y} \end{aligned}$$

Thus, the slope of the line tangent to the graph at the point $(-1,1)$ is

$$m = y' = \frac{16(-1)(1)^2 - 6(-1)((-1)^2+1^1)^2}{6(1)((-1)+1^2)^2 - 16(-1)^2(1)} = \frac{8}{8} = 1$$

and the equation of the tangent line is

$$y - (1) = (1)(x - (-1))$$

or

$$y = x + 2$$

Notes

3. If $x=1$, then $1^2+(y-1)^3=9$, so that

$$\begin{aligned} (y-1)^3 &= 8 \\ y-1 &= 2 \\ y &= 3 \end{aligned}$$

and the tangent line passes through the point $(1,3)$.
Now differentiate both sides of the original equation, getting

$$\begin{aligned} \frac{d}{dx}\left[x^2+(y-x)^3\right] &= \frac{d}{dx}9 \\ 2x + 3(y-x)^2(y'-1) &= 0 \\ 3(y-x)^2(y'-1) &= -2x + 3(y-x)^2 \\ y' &= \frac{-2x+3(y-x)^2}{3(y-x)^2} \end{aligned}$$

Thus the slope of the line tangent to the graph at $(1,3)$ is

$$m = y' = \frac{3(3-1)^2 - 2(1)}{3(3-1)^2} = \frac{10}{12} = \frac{5}{6}$$

and the equation of the tangent is: $y-(3) = \frac{5}{6}(x-1), \quad y = \frac{5}{6}x + \frac{13}{6}$

Notes

Learning Objectives

- To solve applications involving related rates.

Notes

Introduction:

Related rate problems involve generally three variables: an independent variable (often $t =$ time), and two dependent variables.

The goal is to find a formula for the rate of change of one of the independent variables in terms of the rate of change of the other one.

These problems are solved by using a relationship between the variables, differentiating it, and solving for the term you want.

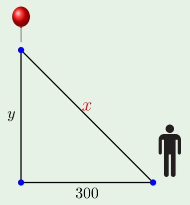
Notes

EXAMPLE 1

A weather balloon is rising vertically at the rate of 5 meters per second. An observer is standing on the ground 300 meters from the point where the balloon was released. At what rate is the distance between the observer and the balloon changing when the balloon is 400 meters high?

Make a drawing.
The independent variable is time t.
Which quantities that change with time are mentioned in the problem?

- We use, $x =$distance from observer to balloon,
- $y =$height of balloon,
- $\frac{dy}{dt}$ is given as $5m/s$,
- $\frac{dx}{dt}$ is the unknown.



Notes

EXAMPLE 1

We need a relationship between x and y:

$$x^2 = y^2 + 300^2 \qquad \text{(Pythagoras)}$$

Differentiate the equation with respect to t:

$$\frac{d}{dt}\left(x^2\right) = \frac{d}{dt}\left(y^2 + 300^2\right)$$
$$2x\frac{dx}{dt} = 2y\frac{dy}{dt}$$

We are looking for dx/dt, so we solve for that:

$$\frac{dx}{dt} = \frac{y}{x}\frac{dy}{dt}$$

Now we need values for x, y and dy/dt. Go back to the problem statement:

- $y = 400$; $dy/dt = 5$; $x = \sqrt{300^2 + 400^2} = 500$

So $dx/dt = (400/500)5 = 4m/s$

Notes

Solving Related Rate Problems

Step 1. Make a sketch.

Step 2. Identify all variables, including those that are given and those to be found.

Step 3. Express all rates as derivatives.

Step 4. Find an equation connecting variables.

Step 5. Differentiate this equation.

Step 6. Solve for the derivative that will give the unknown rate

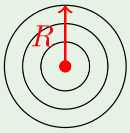
Notes

EXAMPLE 2

A rock is thrown into a still pond and causes a circular ripple. If the radius of the ripple is increasing at 2 feet per second, how fast is the area changing when the radius is 10 feet? (Use $A = \pi R^2$)

Make a drawing.
Which quantities that change with time are mentioned in the problem?

- Let, $R =$radius is given as 10,
- $A =$area,
- $\frac{dR}{dt} = 2$,
- $\frac{dA}{dt}$ is the unknown.



Notes

EXAMPLE 2

We need a relationship between R and A:

$$A = \pi R^2$$

Differentiate this equation with respect to t:

$$\frac{d}{dt}(A) = \frac{d}{dt}\left(\pi R^2\right)$$
$$\frac{dA}{dt} = 2\pi R \frac{dR}{dt}$$
$$\frac{dA}{dt} = 2\pi 10 \cdot 2 = 40\pi$$

Notes

Example 3: Related Rates in Business

Suppose that for a company manufacturing transistor radios, the cost and revenue equations are given by

$$C = 5,000 + 2x$$

and

$$R = 10x - 0.001x^2$$

where the production output in 1 week is x radios.
If production is increasing at the rate of 500 radios per week when production is $2,000$ radios, find the rate of increase in (a) Cost (b) Revenue
These are really two related rates problems, one involving C, x and time t, and one involving $R, x,$ and t.
Differentiate the equations for C and R with respect to time.

$$\frac{dC}{dt} = 2\frac{dx}{dt} = 2 \cdot 500 = 1000$$

Cost is increasing at the rate of $\$1,000$ per week.

Notes

EXAMPLE 3: Related Rates in Business

$$\begin{aligned}\frac{dR}{dt} &= 10\frac{dx}{dt} - 0.002x\frac{dx}{dt}\\ &= 10 \cdot 500 - 0.002 \cdot 2000 \cdot 500\\ &= 3000\end{aligned}$$

Revenue is increasing at the rate of $\$3,000$ per week.

Notes

EXERCISES

1. Assuming that $x = x(t)$ and $y = y(t)$ find the indicated rate, given the other information.
   a. $y = 2x^2 - 1$; $\frac{dx}{dt} = 5$ when $x = 1$; find dy/dt.
   b. $x^2 - 2xy - 14 = y$; $\frac{dx}{dt} = 1$ when $x = 2$; and $y = -2$; find dy/dt.
   c. $x^2 + xy + y^2 = 3$; $\frac{dy}{dt} = 3$ when $x = 2$; and $y = -2$; find dx/dt.

2. A point is moving on the graph of $x^2 + y^2 = 10$. When the point is at $(1, 3)$, its y coordinate is increasing by 3 units per second. How fast is the x coordinate changing at that moment?

3. Suppose Kyle and Daniel were each traveling to their respective homes from college. Kyle traveled south at 65 miles per hour while Daniel traveled east at 60 miles per hour. How fast was the distance between Kyle and Daniel changing after 1 hour?

Notes

EXERCISES

4. Suppose a 50 foot ladder is resting against a building. If the bottom of the ladder is pulled away from the house at a constant rate of 2 feet per second, then how fast is the top of the ladder sliding down the building when the bottom of the ladder is 10 feet from the house?

5. Suppose a company's total weekly profit from the production of x boxes of customized stationary can be modeled by the equation $P(x) = -0.02x^2 + 21x - 1900$. If production is increasing at the rate of 10 boxes per week when production is 150 boxes, find the rate of increase (or decrease) in profit.

6. A screen saver displays the outline of a $3cm$ by $2cm$ rectangle and then expands the rectangle in such a way that the $2cm$ side is expanding at the rate of $4cm/sec$ and the proportions of the rectangle never change. How fast is the area of the rectangle increasing when its dimensions are $12cm$ by $8cm$?

Notes

6. The mathematical model of this problem is a rectangle of varying width x and height y. The independent variable is time t. We are asked to find $\frac{dA}{dt}$ when $x = 12$ and $y = 8$. $\frac{dy}{dt} = 4$ is given. The relationships in the problem are $A = xy$ and $2x = 3y$.

$$\begin{aligned}\frac{dA}{dt} &= \frac{d}{dt}xy \\ &= \frac{dx}{dt}y + x\frac{dy}{dt} \\ &= \frac{dx}{dt}8 + 12 \times 4\end{aligned}$$

Differentiate $2x = 3y$ with respect to t:

$$\begin{aligned}\frac{d}{dt}2x &= \frac{d}{dt}2y \\ 2\frac{dx}{dt} &= 3\frac{dy}{dt} \\ \frac{dx}{dt} &= \frac{3}{2}4 = 6cm/s\end{aligned}$$

$$\frac{dA}{dt} = 6 \times 8 + 12 \times 4 = 96cm/s$$

Notes

Learning Objectives

- Find the relative rate of change of functions.
- Solve applications involving elasticity of demand.

Notes

Relative and Percentage Rates of Change

Remember that $f'(x)$ represents the rate of change of $f(x)$. The relative rate of change is defined as

$$\frac{f'(x)}{f(x)}$$

By the chain rule, this equals the derivative of the logarithm of $f(x)$:

$$\frac{f'(x)}{f(x)} = \frac{d}{dx}\ln f(x)$$

The percentage rate of change of a function $f(x)$ is

$$100 \cdot \frac{f'(x)}{f(x)} = 100 \cdot \frac{d}{dx}\ln f(x)$$

Notes

EXAMPLE 1

Find the relative rate of change of

$$f(x) = 50x - 0.01x^2$$

The derivative of $\ln f(x)$ is

$$\frac{50 - 0.02x}{50x - 0.01x^2}$$

Notes

EXAMPLE 2

A model for the real GDP (gross domestic product expressed in billions of 1996 dollars) from 1995 to 2002 is given by

$$f(t) = 300t + 6,000$$

where t is years since 1990. Find the percentage rate of change of $f(t)$ for $5 \le t \le 12$.
The percentage rate of change of $f(t)$ is given by:

$$\begin{aligned} &= 100 \cdot \frac{d}{dx}\ln(300t + 6,000) \\ &= 100 \cdot \frac{300}{300t + 6,000} \\ &= \frac{30,000}{300t + 6,000} \\ &= \frac{100}{t + 20} \end{aligned}$$

The percentage rate of change in 1995 $(t = 5, \frac{100}{25})$ is 4$\%$.

Notes

Elasticity of Demand

Elasticity of demand describes how a change in the price of a product affects the demand.
Assume that $f(p)$ describes the demand at price p. Then we define

$$\begin{aligned} \text{Elasticity of Demand} &= -\frac{\text{relative rate of change in demand}}{\text{relative rate of change in price}} \\ &= -\frac{\frac{d}{dp}\ln f(p)}{\frac{d}{dp}\ln p} = -\frac{\frac{f'(p)}{f(p)}}{\frac{1}{p}} \\ &= -\frac{pf'(p)}{f(p)} \end{aligned}$$

Notice the minus sign. f and p are always positive, but f' is negative (higher cost means less demand). The minus sign makes the quantity come out positive.

Notes

THEOREM 1: Elasticity of Demand

Given a price-demand equation $x = f(p)$ (that is, we can sell amount x of product at price p), the elasticity of demand is given by the formula

$$E(p) = -\frac{pf'(p)}{f(p)}$$

Notes

Elasticity of Demand Interpretation

$E(p)$	Demand	Interpretation
$E(p) < 1$	Inelastic	Demand is not sensitive to changes in price. A change in price produces a small change in demand.
$E(p) > 1$	Elastic	Demand is sensitive to changes in price. A change in price produces a large change in demand.
$E(p) = 1$	Unit	A change in price produces the same change in demand.

Notes

EXAMPLE 3

For the price-demand equation

$$x = f(p) = 1875 - p^2,$$

determine whether demand is elastic, inelastic or unit for $p = 15, 25$ and 40.

$$E(p) = -\frac{pf'(p)}{f(p)} = -\frac{p(-2p)}{1875 - p^2} = \frac{2p^2}{1875 - p^2}$$

- If $p = 15$, then $E(15) = 0.27 < 1$ demand is inelastic
- If $p = 25$, then $E(25) = 1$ demand has unit elasticity
- If $p = 40$, then $E(40) = 11.64 > 1$ demand is elastic

Notes

Revenue and Elasticity of Demand

- If demand is inelastic, then consumers will tend to continue to buy even if there is a price increase, so a price increase will increase revenue and a price decrease will decrease revenue.
- If demand is elastic, then consumers will be more likely to cut back on purchases if there is a price increase. This means a price increase will decrease revenue and a price decrease will increase revenue.

Notes

Elasticity of Demand for Different Products

Different products have different elasticities. If there are close substitutes for a product, or if the product is a luxury rather than a necessity, the demand tends to be elastic. Examples of products with high elasticities are jewellery, furs, or furniture.

On the other hand, if there are no close substitutes or the product is a necessity, the demand tends to be inelastic. Examples of products with low elasticities are milk, sugar, and light bulbs.

Notes

Summary

- Elastic demand, price increase: total revenue decreases
- Elastic demand, price decrease: total revenue increases
- Inelastic demand, price increase: total revenue increases
- Inelastic demand, price decrease: total revenue decreases
- Unit elastic demand: total revenue does not change

Notes

EXERCISES

1. Find the relative rate of change for the following functions
   a. $f(x) = 20x - 1400$
   b. $f(x) = x + x \ln 2x$
2. Use the price-demand equation to determine whether demand is elastic, inelastic, or has unit elasticity at the indicated values of p.
   a. $x = f(p) = 4000 - 3p^2$ when $p = 15, 25, 35$.
   b. $x = f(p) = 2000 - 0.02p^2$ when $p = 153, 183, 203$.
3. Use the price-demand equation to find the values of p for which demand is elastic and the values for which demand is inelastic: $x = f(p) = \sqrt{289 - 2p}$

Notes

EXERCISES

4. Use the demand equation to find the revenue function. Sketch the graph of the revenue function, and indicate the regions of inelastic demand on the graph.
$x = f(p) = 75(p - 10)^2$
5. Consider the price-demand equation for a certain washing machine: $510 = p + 0.05x$ where p represents the price (in dollars) when x machines are cleaned.
   a. Express the revenue as a function of the price p.
   b. Find the elasticity of demand $E(p)$.
   c. For which values of p is demand elastic? Inelastic?
   d. For which values of p is revenue increasing? Decreasing?
   e. If $p = \$50$ and the price is decreased, will revenue increase or decrease?

Notes