Mathématiques pour SHS

Master Sciences des données et histoire

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5. Graphing and Optimization

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Learning Objectives

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- Use the first derivative test to determine the local extrema of functions.

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- f' is positive at $x = c \Longleftrightarrow$ The line tangent to graph of f at $x = c$ exists and it tilts upward.

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- f' is zero at $x = c \Longleftrightarrow$ The line tangent to graph of f at $x = c$ exists and it is horizontal.

Now talk about behavior on an interval, not just at some particular $x = c$.

DEFINITION

Words: f is increasing on the interval $a < x < b$

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- f' is zero on whole interval $a < x < b \Longleftrightarrow f$ is constant on the whole interval $a < x < b$.

THEOREM 1: Increasing and Decreasing Functions

On the interval (a, b)		
$f'(x)$	$f(x)$	Graph of f
$+$	increasing	rising
$-$	decreasing	falling

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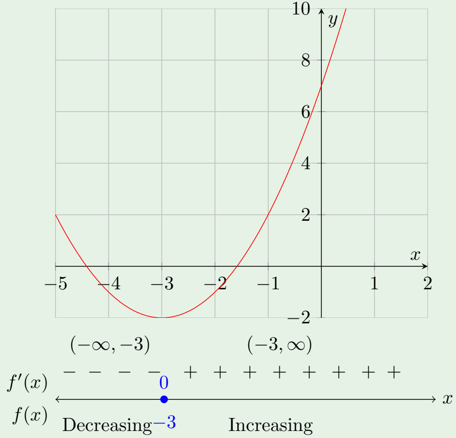
$2x + 6 > 0$ when $2x > -6,$ or $x > -3$.

The graph is rising when $x > -3$.

$2x + 6 < 0$ when $x < -3$, so the graph is falling when $x < -3$.

EXAMPLE 1

A sign chart is helpful:



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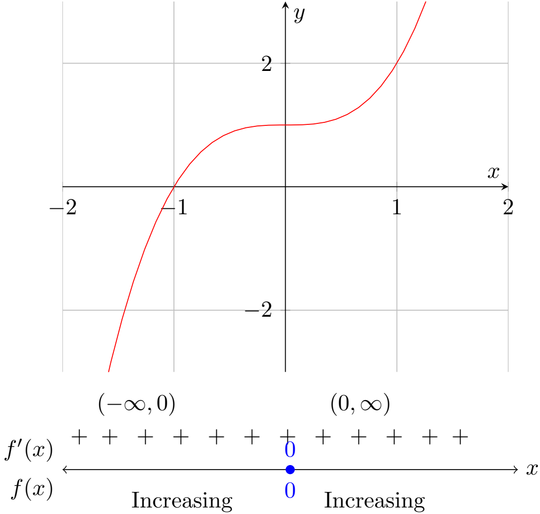
If f is a polynomial, critical values and partition numbers are both the same, namely the solutions of $f'(x) = 0$.

EXAMPLE 2

$f(x) = 1 + x^3$, $f'(x) = 3x^2$. Critical value and partition point at $x = 0$.



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THEOREM **Existence of Local Extrema**

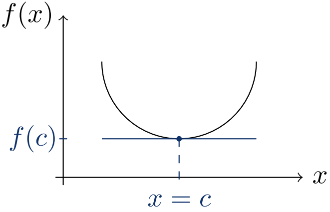
If f is continuous on the interval (a, b), c is a number in (a, b), and $f(c)$ is a local extremum, then either $f'(c) = 0$ or $f'(c)$ does not exist. That is, c is a critical point.

First Derivative Test

Let c be a critical value of f. That is, $f(c)$ is defined, and either $f'(c) = 0$ or $f'(c)$ is not defined. Construct a sign for $f'(x)$ close to and on either side of c.

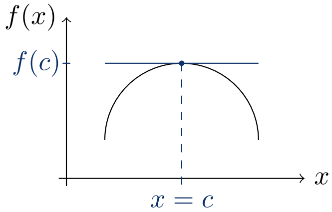
On the interval (a, b)		
$f(x)$ left of c	$f(x)$ right of c	$f(c)$
Decreasing	Increasing	local minimum at c
Increasing	Decreasing	local maximum at c
Decreasing	Decreasing	not an extremum
Increasing	Increasing	not an extremum

$f'(c) = 0$: Horizontal Tangent



$f'(x)$ $\boxed{- \quad - \quad -\, 0 + \quad + \quad +}$

(A) $f(c)$ is a local minimum



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(B) $f(c)$ is a local maximum

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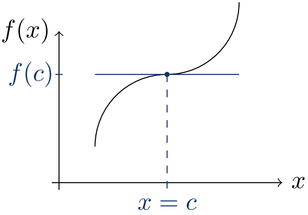
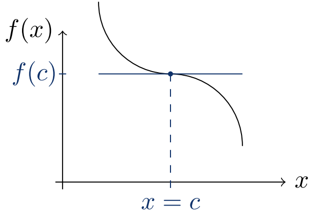


$f'(x)$ $\boxed{+ \; + \;\; +0+ \;\; + \; +}$

(C) $f(c)$ is neither a local max nor a local min



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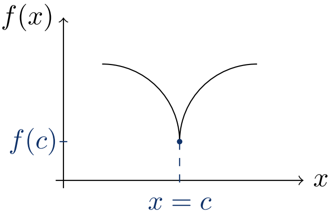
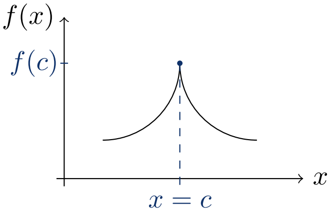
(D) $f(c)$ is neither a local max nor a local min

$f'(c)$ is not defined but $f(c)$ is defined





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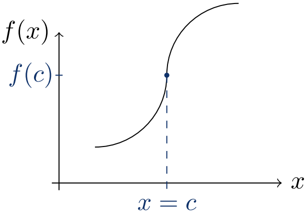

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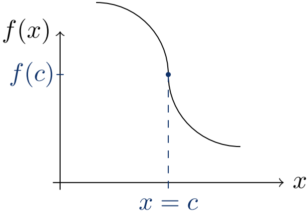

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THEOREM 3 Intercepts and Local Extrema of Polynomial Functions

If

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x_1 + a_0, \quad a_n \neq 0,$$

is an $n^{th}-$degree polynomial, then f has at most n $x-$intercepts and at most $(n-1)$ local extrema.

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THEOREM 3 does not guarantee that every $nth-$degree polynomial has exactly $n-1$ local extrema; it says only that there can never be more than $n-1$ local extrema.

EXERCISES

1. Use a sign graph to determine the intervals where x in increasing or decreasing. Give your answers in interval notation.
   a. $f(x) = 15x^2 - 30x - 60$
   b. $f(x) = 4x^3 - 3x^2$
2. Determine the intervals where $g(x)$ is increasing or decreasing. Identify the critical values of $g(x)$.
   a. $g(x) = \frac{x^3}{3} - x^2 - 15x + 4$
   b. $g(x) = \frac{x^2}{x+4}$
3. Let $f(x) = -x^4 + 50x^2$.
   a. Finds intervals where f is increasing or decreasing. Present the answers three ways: inequality notation and interval notation
   b. Find $x-$coordinates of all local extrema.
   c. Find the $y-$values of the local extrema.
   d. Sketch a graph

EXERCISES

4. Given that $f(x)$ is continuous on $(-\infty, \infty)$, use the information to sketch a graph of $f(x)$.

$$f(4) = 0, f(1) = 9$$
$$f'(1) = 0, f'(x) > 0, \quad \text{on} \quad (1, \infty)$$
$$f'(x) < 0, \quad \text{on} \quad (-\infty, 1)$$

5. Determine the local extrema for the functions in Exercise 2.

5. Graphing and Optimization

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- Solve applications involving the point of diminishing returns.

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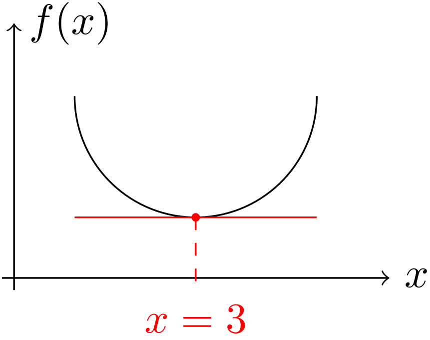
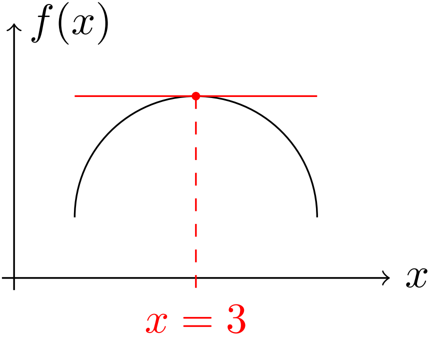


f concave up at $x = 3$



f concave down at $x = 3$

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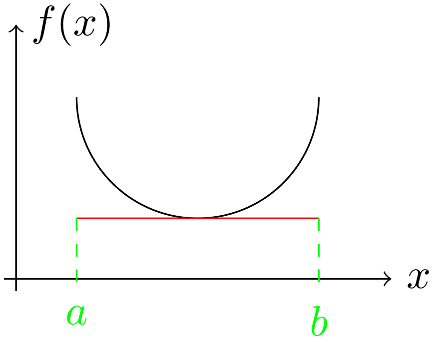

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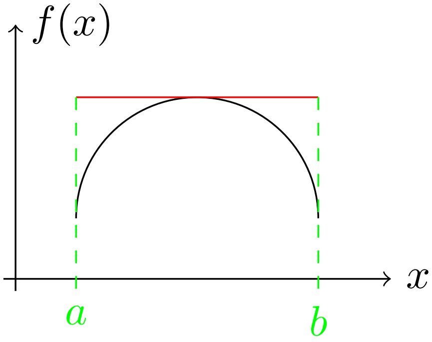
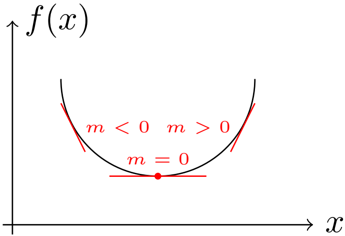
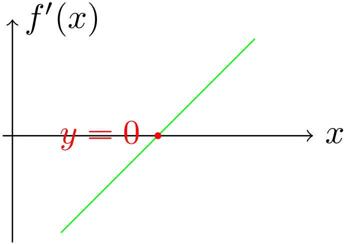


f concave down on (a, b)

Consider relationship between concavity of f and the behavior of f'





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So f' being increasing or decreasing on an interval $a < x < b$ is related to the derivative of f' being positive or negative on the interval $a < x < b$.

This leads us to consider the derivative of f'.

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For $f(x) = -x^4 + 50x^2$ and $f(x) = xe^{-x}$ find $f''(x)$

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Find the intervals where the graph of $f(x) = 2x^5 - 3x^4$ is concave up or concave down.

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EXAMPLE 3

Find the inflection point(s) of $f(x) = 2x^5 - 3x^4$.

Analytical Example

Questions: Given a function

1. Find intervals where function is increasing or decreasing.

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Let $f(x) = xe^{-x}$, Answer questions 1-6.

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   Conculde: The domain of f' is all x. There are no $x = c$ where $f'(c)$ DNE.

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Conclusion: the only $x-$value that will cause $f'(x) = 0$ is $x = 1$.
Conclude: $x = 1$ is the only partition number for $f'(x)$.

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Conclusion: the only $x-$value that will cause $f'(x) = 0$ is $x = 1$.
Conclude: $x = 1$ is the only partition number for $f'(x)$.
Now make a sign chart for $f'(x)$

	$(-\infty, 1)$		$(1, \infty)$	
$f'(x)$	$+ + + + + +$	$f' = 0$	$- - - - -$	
$f(x)$	Increasing	$x = 1$	Deacreasing	x

$$f'(2) = (1-2)e^{-2} : nes \cdot pos = neg$$

$$f'(0) = (1-0)e^{-0} : pos \cdot pos = pos$$

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Conclusion of question 1:

- f is increasing on interval $(-\infty, 1)$, because f' is positive there.
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2 Local max at $x = 1$ because f changes from inc to dec (because f' changes from pos to neg) and because we know that $x = 1$ is a critical value of f.
That is

- $x = 1$ is a partition number for f'
- $f(1)$ exists because domain of f is all real numbers.

No local min!

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4. Strategy:
   - find f''
   - analyze sign of f''
   - use the information about sign of f'' to answer question about concavity of f.

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Observe

$$f''(x) = \underbrace{(x-2)}_{\text{this is a poly, its domain is all x}} \times \underbrace{e^{-x}}_{\text{this always exists for every x}}$$

So the product always exists for every x.

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$$0 = f''(x)$$
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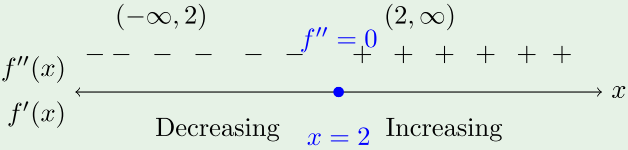
Conclusion: $f''(x)$ has one partition number $x = c = 2$.

EXAMPLE 4



$$f''(1) = (1-2)e^{-1} = neg \cdot pos = neg$$
$$f''(3) = (3-2)e^{-3} = pos \cdot pos = pos$$

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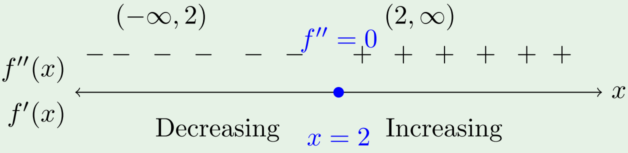

$$f''(1) = (1-2)e^{-1} = neg \cdot pos = neg$$
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Conclusion:

- f is concave up on the interval $(2, \infty)$
- f is concave down on the interval $(-\infty, 2)$

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So there is a point on graph of f at $x = 2$. Conclude there is an inflection point on graph of f at $x = 2$

6 The $y-$value of the inflection point is:

$$f(2) = 2e^{-2} = \frac{2}{e^2}$$

Point of Diminishing Returns

If a company decides to increase spending on advertising, they would expect sales to increase.

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This is also the point where the rate of change has a maximum value. Money spent after this point may increase sales, but at a lower rate. The next example illustrates this concept.

Maximum Rate of Change Example

Currently, a discount appliance store is selling 200 large-screen television sets monthly. If the store invests $\$x$ thousand in an advertising campaign, the ad company estimates that sales will increase to

$$N(x) = 3x^3 - 0.25x^4 + 200, \quad 0 \leq x \leq 9$$

- When is rate of change of sales increasing and when is it decreasing?
- What is the point of diminishing returns and the maximum rate of change of sales?

Maximum Rate of Change Example

The rate of change of sales with respect to advertising expenditures is

$$N'(x) = 9x^2 - x^3 = x^2(9 - x)$$

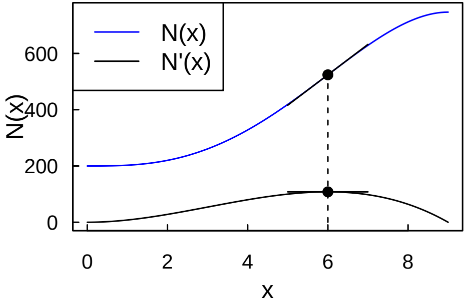
To determine when $N'(x)$ is increasing and decreasing, we find $N''(x)$, the derivative of $N'(x)$:

$$N''(x) = 18x - 3x^2 = 3x(6 - x)$$

5. Graphing and Optimization

5-2 Second Derivative and Graphs

```
N <- function(x){3*x^3-0.25*x^4+200}
Np <- function(x){9*x^2-x^3}; tgt <- function(x) {108*x-124}
curve(N,0,9,col="blue", ylim=c(0,750)); curve(Np,0,9,add=T)
segments(5,Np(6),7,Np(6)); segments(6,0,6,N(6), lty=2)
segments(5, tgt(5), 7, tgt(7))
points(6,Np(6), pch=16);points(6,N(6), pch=16)
legend("topleft", legend=c("N(x)","N'(x)"),
       col=c("blue","black"), lty=1)
```



EXERCISES

1. Find the interval where the graph of f is concave up and concave down. Identify all infection points of $f(x)$.

a. $f(x) = x^3 - 3x^2 + 2x - 1$

b. $f(x) = e^{-3x^2}$

c. $f(x) = \frac{x}{2x-1}$

2. A company estimates that it will sell $N(x)$ units of a product after spending $\$x$ thousand on advertising, as given by

$$N(x) = -0.25x^4 + 13x^3 - 180x^2 + 10,000, \quad 15 \leq x \leq 24$$

- When is rate of change of sales increasing and when is it decreasing?
- What is the point of diminishing returns and the maximum rate of change of sales?
- Graph N and N' on the same coordinate system

5. Graphing and Optimization

Learning Objectives

- Use the graphing strategy to sketch the graphs of functions.

PROCEDURE **Graphing Strategy**

Step 1 Analyze $f(x)$

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1. Find the partition numbers and critical values of $f'(x)$.
2. Construct a sign chart for $f'(x)$.

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Step 2 Analyze $f'(x)$

1. Find the partition numbers and critical values of $f'(x)$.
2. Construct a sign chart for $f'(x)$.
3. Determine the intervals where f is increasing and decreasing.

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1. Find the partition numbers for $f''(x)$.
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3. Determine the intervals where f is concave up or down.

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1. Draw asymptotes, local max/min, and inflection points.

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Step 4 Sketch the graph of f

1. Draw asymptotes, local max/min, and inflection points.
2. Plot additional points as needed and complete the sketch.

EXAMPLE 1

Apply the graphing strategy to sketch the graph of $f(x) = x^3 - 3x^2$.

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1. Domain: the domain of f is all $x-$values (poly).

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1. Domain: the domain of f is all $x-$values (poly).
2. y intercept: if $x = 0$, then $f(0) = 0^3 - 3(0^2) = 0$ is the $y-$intercept
   x intercept: if $y = 0$, then $x^3 - 3x^2 = x^2(x-3) = 0$ so that $x = 0$ and $x = 3$ are the $x-$intercepts.

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   x intercept: if $y = 0$, then $x^3 - 3x^2 = x^2(x-3) = 0$ so that $x = 0$ and $x = 3$ are the $x-$intercepts.
3. There are no vertical or horizontal asymptotes since f is a polynomial.

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Step 2 Analyze $f'(x)$. $f'(x) = 3x^2 - 6x = 3x(x-2)$

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   Partition numbers for $f'(x)$: $x = 0$ and $x = 2$.
2. Sign chart for $f'(x)$:

	$(-\infty, 0)$		$(0, 2)$		$(2, \infty)$
$f'(x)$	$+ + + +$	0	$- - -$	0	$+ + + + +$
$f(x)$	Inc	$x = 0$	Dec	$x = 2$	Inc

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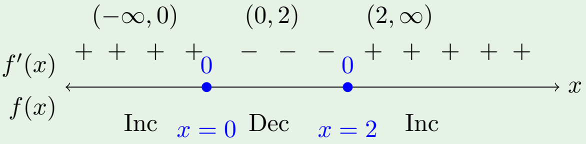

3. f increases on $(-\infty, 0)$ and $(2, \infty)$ and decreases on $(0, 2)$.

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$f'(x)$	$+ + + +$	0	$- - -$	0	$+ + + + +$	
	$\longleftarrow$	$\bullet$	$\longrightarrow$	$\bullet$	$\longrightarrow$	x
$f(x)$	Inc	$x = 0$	Dec	$x = 2$	Inc	

3. f increases on $(-\infty, 0)$ and $(2, \infty)$ and decreases on $(0, 2)$.
4. f has a local max at $x = 0$, $y = 0$. f has a local min at $x = 2$, $y = -4$

EXAMPLE 1

Step 3 Analyze $f''(x)$. $f''(x) = 6x - 6 = 6(x - 1)$

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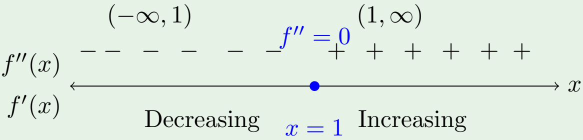

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	$(-\infty, 1)$	$x = 1$	$(1, \infty)$
$f''(x)$	$- - - - - -$	$f'' = 0$	$+ + + + + +$
$f'(x)$	Decreasing	$x = 1$	Increasing

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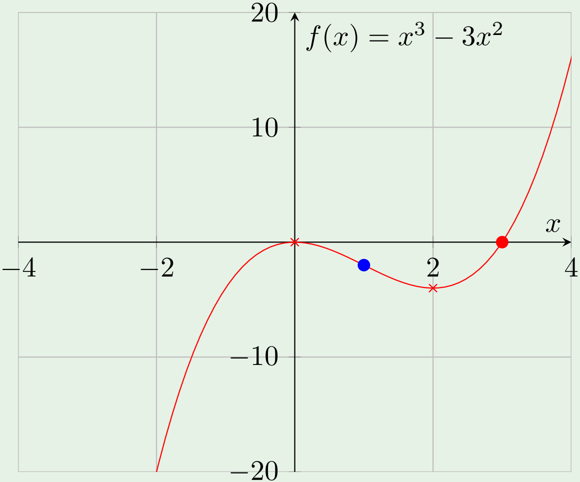
4. f has an inflection point at $x = 1$, $y = -2$

EXAMPLE 1

Step 4 Sketch the graph of f



EXAMPLE 2

If x items are produced in one day, the cost per day is

$$C(x) = x^2 + 2x + 2000$$

and the average cost per unit is $C(x)/x$.
Use the graphing strategy to analyze the average cost function.

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2. y intercept: None

   x intercept: None
3. H.A.: None

   V.A.: The line $x = 0$ is a vertical asymptote ($C(0) \neq 0$).

EXAMPLE 2

Oblique Asymptotes: If a graph approaches a line that is neither horizontal nor vertical as x approaches ∞ or $-\infty$, that line is called an oblique asymptote

$$\bar{C}(x) = \frac{C(x)}{x} = \frac{x^2 + 2x + 2000}{x} = x + 2 + \frac{2000}{x}$$

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If x is a large positive number, then $2000/x$ is very small and the graph of $\bar{C}(x)$ approaches the line $y = x + 2$.
This is the oblique asymptote.

EXAMPLE 2

Step 2 Analyze $\bar{C}'(x)$.

$$\bar{C}'(x) = \frac{(2x+2)x - (x^2+2x+2000)(1)}{x^2} = \frac{x^2-2000}{x^2}$$

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1. Critical values of $\bar{C}(x) : x = \sqrt{2000} \approx 44.72$.
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	$(0, 44.72)$		$(44.72, \infty)$	
$\bar{C}'(x)$	$- - \; - \; - \; - \; -$	0	$+ \; + \; + \; + \; + \; +$	
$\bar{C}(x)$	Decreasing	$x = 44.72$	Increasing	x

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3. If we test values to the left and right of the critical point, we find that $\bar{C}$ is decreasing on $(0, \sqrt{2000})$, and increasing on $(\sqrt{2000}, \infty)$

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4. $\bar{C}$ has a local min at $x = \sqrt{2000}$, $y = 91.44$

EXAMPLE 2

Step 3 Analyze $\bar{C}''(x)$.

$$\bar{C}''(x) = \frac{2x(x^2) - (x^2 - 2000)(2x)}{x^4} = \frac{4000}{x^3}$$

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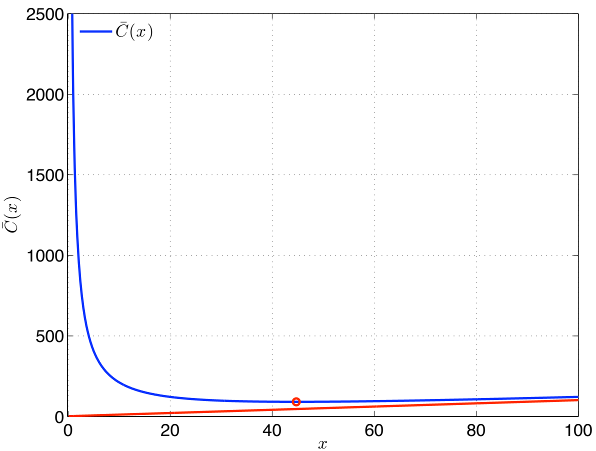
$$\bar{C}''(x) = \frac{2x(x^2) - (x^2 - 2000)(2x)}{x^4} = \frac{4000}{x^3}$$

Since this is positive for all positive x, the graph of the average cost function is concave up on $(0, \infty)$

EXAMPLE 2

Step 4 Sketch the graph of $\bar{C}$.



EXERCISES

1. Summarize the pertinent information obtained by applying the graphing strategy and sketch the graph of $y = f(x)$.

   a. $f(x) = \frac{x^2}{x+1}$

   b. $f(x) = \frac{2x^2-3x}{x-2}$

2. Nicole owns a company that makes luxurious velvet robes. Her total cost to make x robes can be modeled by the function

$$C(x) = 1500 + 3x^2, \quad x > 0.$$

   a. Find the average cost function.

   b. How many robes must be produced for the average cost to be minimized?

   c. What is the minimum average cost?

5. Graphing and Optimization

Learning Objectives

- Find the absolute maxima and absolute minima of functions.

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THEOREM 1

If a function f is continuous on closed interval $[a, b]$, then f is guaranteed to have an absolute max and an absolute min on that interval.

THEOREM 2

The only place where an abs max or min can ever occur (if they occur at all) is at the $x-$values that are

- critical values
- endpoints of the domain

Suppose that the domain of a function f is a closed interval $[a, b]$.
and suppose that it is known that f is continuous on $[a, b]$.

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Theorem 1 guarantees that there will be both an absolute maximum and an absolute minimum on the interval $[a, b]$.

and Theorem 2 tells us where (at what $x-$values) the absolute max and min have to be found.

- at x values that are critical values of f
- at x values that are endpoints ($x = a, x = b$).

This give us the idea for a strategy:

PROCEDURE Finding Absolute Extrema on a Closed Interval

Used for finding the absolute extrema for a function f that is continuous on a closed interval $[a, b]$.

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Step 5 Find the correspond $y-$values.

Step 6 Identify the largest $y-$value as the abs max and the smallest $y-$value as the absolute min. State your conclusion clearly

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Are there any $x-$values that cause $f'(x)$ to not exist? $f'(x)$ is a polynomial so $f'(x)$ exists for all x.
Are there any $x-$values that cause $f'(x) = 0$?Set $f'(x) = 0$ and solve for x.

$$4x^3 - 12x = 0$$

Identify common factor $4x$ and rewrite to highlight the common factor.

$$4x \cdot x^2 - 4x \cdot 3 = 0$$

Now factor out the $4x$: $4x(x^2 - 3) = 0$

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Step 3 Factor some more

$$4x(x - \sqrt{3})(x + \sqrt{3}) = 0$$

Solution: $x = 0, x = -\sqrt{3}, x = \sqrt{3}$ these are the partition numbers for $f'(x)$ because they cause $f'(x) = 0$.

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Observe that $f(x)$ exists at all three of these partition numbers for f' (because f is a poly, so its domain is all real numbers).

So the three $x-$values $x=0, x=-\sqrt{3}, x=\sqrt{3}$ all satisfy

- $f'(x)=0$
- $f(x)$ exists

So these three $x-$values are the critical values for f.

EXAMPLE 1

Step 4-6 List of important $x - values$

Important $x-$values	Corresponding $y-$values
$x = -3$	$y = 32$
$x = -\sqrt{3}$	$y = -4$
$x = 0$	$y = 5$
$x = \sqrt{3}$	$y = -4$
$x = 2$	$y = -3$

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Conclusion:

- The absolute max is $y = 32$ and it occurs at $x = -3$
- The absolute min is $y = -4$ and it occurs at $x = -\sqrt{3}$ and $x = \sqrt{3}$

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Step 3 Critical values of f:
$x = 0$ and $x = \sqrt{3}$
$\cancel{x = -\sqrt{3}}$ not in the interval $[-1, 2]$

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Conclusion:

- The absolute max is $y = 5$ and it occurs at $x = 0$
- The absolute min is $y = -4$ and it occurs at $x = \sqrt{3}$

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We cannot use the closed interval procedure!
So what do we do?

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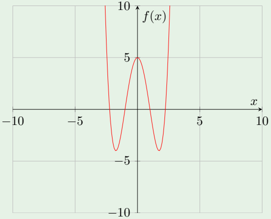
A variety of math technique have to be used, depending on the problem.
Observe f is even degree polynomial with positive leading coefficient. So both ends go up.



So graph will have absolute min, but will not have an absolute max.

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THEOREM 1 tells us that the only places where abs max or min can occur at

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We don't have any endpoints in this example, so the abs max or min must occur at critical values.
From previous example, we know that the critical values of f are:
$x = 0, x = -\sqrt{3}, x = \sqrt{3}$.
So it must be that $y = -4$ is the abs min (it occurs at $x = -\sqrt{3}$ and $x = \sqrt{3}$). No abs max!

Second-Derivative Test

Let c be a critical value of $f(x)$.

$f'(c)$	$f''(c)$	Graph of f is	$f(c)$
0	$+$	Concave up	Local min
0	$-$	Concave do,wn	Local max
0	0	Concave up	Test fails

EXAMPLE 4

Find the local maximum and minimum values of $f(x) = x^3 - 6x^2$ on $[-1, 7]$.

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Find the local maximum and minimum values of $f(x) = x^3 - 6x^2$ on $[-1, 7]$.

$$f'(x) = 3x^2 - 12x = 3x(x-4)$$
$$f''(x) = 6x - 12 = 6(x-2)$$

Critical values: $x = 0$ and $x = 4$

$$f''(0) = -12, \text{hence} f(0) \text{ local max}$$
$$f''(4) = 12, \text{hence} f(4) \text{ local min}$$

THEOREM 3: Second-Derivative Test for Absolute Extremum

Let f be continuous on interval I with only one critical value c in I.

- If $f'(c) = 0$ and $f''(c) > 0$, then $f(c)$ is the absolute minimum of f on I.
- If $f'(c) = 0$ and $f''(c) < 0$, then $f(c)$ is the absolute maximum of f on I.

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The second-derivative test does not apply if $f''(c) = 0$ or if $f''(c)$ is not defined. The first-derivative test must be used.

EXAMPLE 5

Find the absolute minimum value of $f(x) = x + \frac{4}{x}$ on $(0, \infty)$.

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Find the absolute minimum value of $f(x) = x + \frac{4}{x}$ on $(0, \infty)$.

$$f'(x) = 1 - \frac{4}{x^2} = \frac{x^2 - 4}{x^2} = \frac{(x-2)(x+2)}{x^2}$$

$$f''(x) = \frac{8}{x^3}$$

The only critical value in the interval $(0, \infty)$ is $x = 2$. Since $f''(2) = 1 > 0$, $f(2)$ is the abs min value of f on $(0, \infty)$

EXERCISES

1. Use the second derivative test to find the local extrema for $f(x) = 2x^3 - 4x^2 - 10$
2. Let $f(x) = 20 - 4x - \frac{250}{x^2}$. Find all absolute extrema on the interval $(0, \infty)$
3. Find the absolute maxima and absolute minima, if they exist, for the function $f(x) = \frac{x^3}{3} - x^2 + 4$ on the given intervals.
   a. $[-4, 0]$
   b. $[-4, 3]$

5. Graphing and Optimization

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Possible Complications:

- problems may be word problems.

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- problems might involve more than one variable.

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- problems may be word problems.
- domain might not be closed intervals.
- domain might not even be specified (you will have to figure it out).
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The techniques used to solve optimization problems are best illustrated through examples. Let's begin with some examples.

EXAMPLE 1

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Substitute into equation II: $10x + 25\frac{9000}{x} = S$
This describes a function S of the variable x. In function notation

$$S(x) = 10x + 25\frac{9000}{x}$$

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Goal: Find absolute min of $S(x)$ on the interval $(0, \infty)$.

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Start by finding partition numbers of $S'(x)$ that is $x-$values where $S' = 0$ or S' DNE.

$$\begin{aligned} S(x) &= 10x + 25\frac{9000}{x} = 10x + 25(9000)x^{-1} \\ S'(x) &= \frac{d}{dx}\left(10x + 25(9000)x^{-1}\right) \\ &= 10 + 25(9000)(-1)x^{-2} \\ &= 10 - \frac{25(9000)}{x^2} \end{aligned}$$

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Are there any $x-$values that cause $S'(x)=0$?
Set $S'(x)=0$ and solve for x.

EXAMPLE 1

$$\begin{aligned} 10 - \frac{25(9000)}{x^2} &= 0 \\ 10 &= \frac{25(9000)}{x^2} \\ 10x^2 &= 25(9000) \\ x^2 &= 25(900) \\ x &= \sqrt{25(900)} = \sqrt{25}\sqrt{900} \\ &= 5 \cdot 30 = 150 \end{aligned}$$

So $x = 150$ is a partition number for S' because $S'(150) = 0$.

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Is $x = 150$ a critical value for S?
Does $S(150)$ exists?
$S(150) = 10(150) + \frac{25(900)}{150}$, this exists!
So $x = 150$ is a partition number for $S'(x)$ has property that $S(150)$ exists.
So $x = 150$ is a critical value for S. This must be the place where the min occurs.

EXAMPLE 1

Study sign of $S'(x)$.

	$(0, 150)$	$x = 150$	$(150, \infty)$	
$S'(x)$	$- - \ - \ - \ - \ -$	0	$+ \ + \ + \ + \ + \ +$	x
$S(x)$	Decreasing		Increasing	

So $x = 150$ is the location of the absolute min.
We still need to find y. Must satisfy

$$xy = 9000$$
$$y = \frac{9000}{x}$$
$$y = \frac{9000}{150} = 60$$

$(x, y) = (150, 60)$

EXAMPLE 2

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From the area equation solve for L and substitute that value of L into the perimeter equation to get an equation in one unknown:

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We wish to minimize $P(W)$, so we take the derivative and look at the critical values.

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$$
\begin{aligned}
P'(W) &= \frac{d}{dW}\left(\frac{450}{W} + 2W\right) = \frac{-450}{W^2} + 2 \\
&= \frac{2W^2 - 450}{W^2} = \frac{2(W^2 - 225)}{W^2} = \frac{2(W - 15)(W + 15)}{W^2}
\end{aligned}
$$

There is a critical value at $W = 15$. (Disregard $W = -15$ since the width cannot be negative).

5. Graphing and Optimization

5-6 Optimization

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There is a critical value at $W = 15$. (Disregard $W = \breve{}15$ since the width cannot be negative).

$$P''(W) = \frac{900}{W^3}$$

$P''(15) > 0$, so this is a local minimum and since $W = 15$ is the only critical value, then $P(15) = \frac{450}{15} + 2 \cdot 15 = \$60$ must be the absolute minimum value of $P(W)$. The least perimeter occurs when $W = 15$.
For this value $L = \frac{225}{15} = 15$, so the shape is a square of side 15 meters, with minimum perimeter of 60.

PROCEDURE Strategy for Solving Optimization Problems

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Step 4 Use the solution to the mathematical model to answer all the questions asked in the problem.

EXAMPLE 3

A company manufactures and sells x television sets per month. The monthly cost and price-demand equations are:

$$\begin{aligned} C(x) &= 60,000 + 60x \\ p(x) &= 200\breve{\ }x/50, \qquad \text{for } 0 \leq x \leq 6,000 \end{aligned}$$

a Find the production level that will maximize the revenue, the maximum revenue, and the price that the company needs to charge at that level.

b Find the production level that will maximize the profit, the maximum profit, and the price that the company needs to charge at that level.

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a The monthly revenue is

$$R(x) = xp(x) = x(200\breve{\ }x/50) = 200x - \frac{x^2}{50}$$

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Differentiate and set to zero:

$$R'(x) = 200 - \frac{x}{25} = 0$$
$$x = 5000$$

EXAMPLE 3

a Use the second-derivative test for absolute extrema:

$$R''(x) = -\frac{1}{25} < 0, \quad \text{for all} x$$

Since $x = 5000$ is the only critical value and $R''(x) < 0$,

$$\text{Max } R(x) = R(5000) = \$500,000$$

When the demand is $x = 5000$, the price is

$$p(5000) = \$100$$

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b $\text{Profit} = \text{Revenue˘Cost}$

$$P(x) = 200x - \frac{x^2}{50} - (60000 + 60x) = -\frac{x^2}{50} + 140x - 60000$$

$$P'(x) = \frac{-x}{25} + 140 = 0$$

$$x = 3500$$

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Summary:
The maximum revenue of $\$500,000$ is achieved at a production level of 5000 sets per month, which are sold at $\$100$ each. (The profit is $P(5000) = \$140,000$.)

The maximum profit of $\$185,000$ is achieved at a production level of 3500 sets per month, which are sold at $\$130$ each. (The revenue is $R(3500) = \$455,000$).

EXAMPLE 4: **Inventory Control**

A pharmacy has a uniform annual demand for 200 bottles of a certain antibiotic. It costs $\$5$ per year for a storage place for one bottle, and $\$40$ to place an order.

How many times during the year should the pharmacy order the antibiotic in order to minimize total cost?

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Les $x =$ number of bottles per order, and $y =$ number of orders.
The total annual cost is $C = 40y + 5x$.
In order to write the total cost C as a function of one variable, we must find a relationship between x and y.

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The total annual cost is $C = 40y + 5x$.
In order to write the total cost C as a function of one variable, we must find a relationship between x and y.
The total number of bottles is $xy = 200$, so $y = \frac{200}{x}$.

EXAMPLE 4: **Inventory Control**

Certainly, x must be at least 1 and cannot exceed 200. We must solve the following equation:

$$\text{Minimize } C(x) = \frac{8000}{x} + 5x \qquad 1 \le x \le 200$$
$$C'(x) = -\frac{8000}{x^2} + 5 = 0$$
$$x = 40$$
$$C''(x) = \frac{8000}{x^3} > 0 \quad \text{for } x \in (1, 200)$$

Therefore,

$$\text{Min } C(x) = C(40) = \frac{8000}{40} + 5 \cdot 40 = 400$$
$$y = \frac{200}{40} = 5$$

The pharmacy will minimize its total cost by ordering 40 bottles five times during the year.

EXERCISES

1. Find two positive numbers x, y such that

- the sum $2x + y = 900$.
- the product $A = xy$ is maximized.

2. A farmer needs to build a fence to make a rectangular yard next to an adjacent pasture. He only needs to fence 3 sides because the 4th side already has a fence. He has 900 feet of fence to use.
What dimensions give the largest yard?

3. Katie is a seamstress who makes wedding dresses. Her monthly cost and revenue functions when making x wedding dresses can be modeled approximately by $C(x) = 200 + 150x$ and $R(x) = 700x - 35x^2$, where $0 \leq x \leq 15$

- How many dresses should Katie make each month to maximize revenue?
- How many dresses should Katie make each month to maximize profit?
- Are the values from parts a and b the same? If not, explain why they may be different.