

# Mathématiques pour SHS

Master Sciences des données et histoire

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## 6. Integration

6-1 Antiderivative and Indefinite Integrals

### Learning Objectives

- Use indefinite integral formulas and properties to find antiderivatives.
- Solve applications that require finding antiderivatives.

## 6. Integration

- 1 6-1 Antiderivative and Indefinite Integrals
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## 6. Integration

6-1 Antiderivative and Indefinite Integrals

### DEFINITION: Antiderivative

**Words:**  $F$  is an antiderivative of  $f$ .

**Meaning:**  $f$  is the derivative of  $F$

$$\begin{aligned} f &= F' \\ F' &= f \end{aligned}$$

## 6. Integration

### 6-1 Antiderivative and Indefinite Integrals

#### EXAMPLE 1

$F(x) = \frac{x^3}{3}$  is an antiderivative of  $f(x) = x^2$ . Check:

$$F'(x) = \frac{d}{dx} F(x) = \frac{d}{dx} \left( \frac{x^3}{3} \right) = \frac{1}{3} \frac{d}{dx} x^3 = \frac{1}{3} 3x^2 = x^2 = f(x)$$

#### EXAMPLE 2

$G(x) = \frac{x^3}{3} + 17$  is an antiderivative of  $f(x) = x^2$ . Check:

$$G'(x) = \frac{d}{dx} G(x) = \frac{d}{dx} \left( \frac{x^3}{3} + 17 \right) = \frac{1}{3} \frac{d}{dx} x^3 + \frac{d}{dx} 17 = \frac{1}{3} 3x^2 + 0 = x^2 = f(x)$$

So there is **more than one** antiderivative of  $f(x)$ . That is, given any antiderivative  $F(x)$  for  $f(x)$ , then we can make lots of other antiderivatives. Any function of the form  $F(x) + c$  will also be an antiderivative of  $f(x)$ .

## 6. Integration

### 6-1 Antiderivative and Indefinite Integrals

#### THEOREM 1 Antiderivatives

These are the only antiderivates of  $f(x)$ . That is, if one antiderivative of  $f(x)$  is  $F(x)$ , then all the other antiderivates of  $f(x)$  are of the form

$$F(x) + c$$

(where  $c$  is a constant that can be any real number)

## 6. Integration

### 6-1 Antiderivative and Indefinite Integrals

#### Indefinite Integral

Let  $f(x)$  be a function. The family of all functions that are antiderivatives of  $f(x)$  is called the **indefinite integral** and has the symbol

$$\int f(x) dx$$

The symbol  $\int$  is called an **integral sign**, and the function  $f(x)$  is called **integrand**. The symbol  $dx$  indicates that antidifferentiation is performed with respect to the variable  $x$ . By the previous theorem, if  $F(x)$  is any antiderivative of  $f(x)$ , then

$$\int f(x) dx = F(x) + C$$

The arbitrary constant  $C$  is called the **constant of integration**.

## 6. Integration

### 6-1 Antiderivative and Indefinite Integrals

#### EXAMPLE 3

Find the indefinite integral of  $f(x) = x^2$ . Check:

$$\int f(x) dx = \int x^2 dx = \frac{x^3}{3} + C,$$

because

$$\frac{d}{dx} \left( \frac{x^3}{3} + C \right) = x^2$$

## 6. Integration

### 6-1 Antiderivative and Indefinite Integrals

#### Formulas and Properties

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad \text{valid when } n \neq -1$$

$$\int e^x dx = e^x + C$$

$$\int \frac{1}{x} dx = \ln |x| + C$$

$$\int k f(x) dx = k \int f(x) dx$$

$$\int [f(x) \pm g(x)] dx = \int f(x) dx \pm \int g(x) dx$$

## 6. Integration

### 6-1 Antiderivative and Indefinite Integrals

#### Formulas and Properties

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad \text{valid when } n \neq -1$$

Check by finding the derivative

$$\begin{aligned} \frac{d}{dx} \left( \frac{x^{n+1}}{n+1} + C \right) &= \frac{d}{dx} \left( \frac{x^{n+1}}{n+1} \right) + \frac{d}{dx} C \\ &= \frac{1}{n+1} \frac{d}{dx} x^{n+1} + 0 \\ &= \frac{1}{n+1} (n+1) x^{(n+1)-1} \\ &= x^n \end{aligned}$$

## 6. Integration

### 6-1 Antiderivative and Indefinite Integrals

#### EXAMPLE 4

$$\int x^8 dx = \frac{x^{8+1}}{8+1} + C = \frac{x^9}{9} + C$$

Check:

$$\frac{d}{dx} \left( \frac{x^9}{9} + C \right) = \frac{1}{9} 9x^8 + 0 = x^8$$

#### EXAMPLE 5

$$\int x^{-4} dx = \frac{x^{-4+1}}{-4+1} + C = \frac{x^{-3}}{-3} + C$$

Check:

$$\frac{d}{dx} \left( \frac{x^{-3}}{-3} + C \right) = -3 \frac{x^{-3-1}}{-3} = x^{-4}$$

## 6. Integration

### 6-1 Antiderivative and Indefinite Integrals

#### EXAMPLE 6

$$\int x^{2/3} dx = \frac{x^{\frac{2}{3}+1}}{\frac{2}{3}+1} + C = \frac{x^{\frac{10}{3}}}{\frac{10}{3}} + C = \frac{3}{10} x^{\frac{10}{3}} + C$$

#### EXAMPLE 7

$$\int x dx = \frac{x^{1+1}}{1+1} + C = \frac{x^2}{2} + C$$

#### EXAMPLE 8

$$\int 1 dx = \int x^0 dx = \frac{x^{0+1}}{0+1} + C = x + C$$

## 6. Integration

### 6-1 Antiderivative and Indefinite Integrals

#### Application

In spite of the prediction of a paperless computerized office, paper and paperboard production in the United States has steadily increased. In 1990 the production was 80.3 million short tons, and since 1970 production has been growing at a rate given by

$$f'(x) = 0.048x + 0.95,$$

where  $x$  is years after 1970.

Find  $f(x)$  and the production levels in 1970 and 2000.

## 6. Integration

### 6-1 Antiderivative and Indefinite Integrals

#### Application

We need the integral of  $f'(x)$

$$f(x) = \int f'(x)dx = \int (0.048x + 0.95)dx = 0.048 \frac{x^2}{2} + 0.95x + C = 0.24x^2 + 0.95x + C$$

Noting that  $f(20) = 80.3$ , we calculate

$$80.3 = 0.24(20)^2 + 0.95 \cdot 20 + C$$

$$80.3 = 28.6 + C$$

$$C = 51.7$$

So  $f(x) = 0.24x^2 + 0.95x + 51.7$

The years 1970 and 2000 correspond to  $x = 0$  and  $x = 30$ .

$$f(0) = 51.7$$

$$\begin{aligned} f(30) &= 0.24(30)^2 + 0.95 \cdot 30 + 51.7 \\ &= 101.8 \end{aligned}$$

The production was 51.7 short tons in 1970, and 101.8 short tons in 2000.

## 6. Integration

### 6-1 Antiderivative and Indefinite Integrals

#### EXERCISES

1. Find each indefinite integral.

$$\begin{array}{llll} a. \int 5dx & b. \int edx & c. \int 4x^3dx & d. \int tdt \\ e. \int u^{21}du & f. \int -7\sqrt{x}dx & g. \int \frac{2}{w}dw & h. \int -e^x \end{array}$$

2. Find the antiderivatives for each derivative.

$$a. \frac{dy}{dx} = 12 - x^5 \quad b. \frac{dy}{dt} = 1 - 2t + t^3 \quad c. \frac{du}{dv} = \frac{2}{v} + \frac{v}{2}$$

3. Find each indefinite integral.

$$a. \int 3x(x^2 - 1)dx \quad \int \left( \frac{x^2}{3} - \frac{3}{x^2} + \frac{3}{x} \right) dx \quad c. \int \frac{t^2 - t}{t} dt$$

4. Find the antiderivative of  $\frac{dy}{dx} = e^x - 1$  if  $y(0) = 5$ .

5. The marginal profit from the sales of  $x$  items is given by  $P'(x) = -0.01x + 450$ . Find  $P(x)$  if  $P(100) = 2500$ .

## 6. Integration

### 6-2 Integration by Substitution

#### Learning Objectives

- Integrate expressions by reversing the chain rule.
- Integrate expressions using substitution.

## 6. Integration

### 6-2 Integration by Substitution

#### Reversing the Chain Rule

Recall the chain rule:

$$\frac{d}{dx} f[g(x)] = f'[g(x)]g'(x)$$

Reading it backwards, this implies that

$$\int f'[g(x)]g'(x)dx = \int \frac{d}{dx} f[g(x)]dx = f[g(x)] + C$$

#### Special Cases

$$\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + C, \quad n \neq -1$$

$$\int e^{f(x)} f'(x)dx = e^{f(x)} + C$$

$$\int \frac{f'(x)}{f(x)}dx = \ln |f(x)| + C$$

## 6. Integration

### 6-2 Integration by Substitution

#### DEFINITION Differentials

If  $y = f(x)$  is a differentiable function, then

- The **differential**  $dx$  of the independent variable  $x$  is any arbitrary real number.
- The **differential**  $dy$  of the dependent variable  $y$  is defined as

$$dy = f'(x)dx$$

## 6. Integration

### 6-2 Integration by Substitution

#### EXAMPLE 1

$$\int (x^5 - 2)^3 (5x^4)dx$$

Note that the derivative of  $x^5 - 2 = 5x^4$ , the integral appears to be in the chain rule form

$$\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + C, \quad n \neq -1$$

with  $f(x) = x^5 - 2$  and  $n = 3$

It follows that

$$\int (x^5 - 2)^3 (5x^4)dx = \frac{(x^5 - 2)^4}{4} + C$$

## 6. Integration

### 6-2 Integration by Substitution

#### EXAMPLE 2

- if  $y = f(x) = x^5 - 2$ , then

$$dy = f'(x)dx = 5x^4 dx$$

- if  $y = f(x) = e^{5x}$ , then

$$dy = f'(x)dx = 5e^{5x} dx$$

- if  $y = f(x) = \ln(3x - 5)$ , then

$$dy = f'(x)dx = \frac{3}{3x - 5} dx$$

## 6. Integration

### 6-2 Integration by Substitution

#### EXAMPLE 3

Find  $\int (x^2 + 1)^5 2x dx$  For our substitution, let  $u = x^2 + 1$ , then  $\frac{du}{dx} = 2x$ , use it to build  $dx$

$$dx = \frac{du}{2x}$$

The integral becomes

$$\int (x^2 + 1)^5 2x dx = \int u^5 2x \frac{du}{2x} = \int u^5 du = \frac{u^6}{6} + C$$

Substitute  $u = x^2 + 1$

$$\int (x^2 + 1)^5 2x dx = \frac{(x^2 + 1)^6}{6} + C$$

## 6. Integration

### 6-2 Integration by Substitution

#### General Indefinite Integral Formulas

$$\int u^n du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$$

$$\int e^u du = e^u + C$$

$$\int \frac{1}{u} du = \ln |u| + C$$

## 6. Integration

### 6-2 Integration by Substitution

#### PROCEDURE Integration by Substitution

- Step 1** Select a substitution that appears to simplify the integrand. In particular, try to select  $u$  so that  $du$  is a factor of the integrand.
- Step 2** Express the integrand entirely in terms of  $u$  and  $du$ , completely eliminating the original variable.
- Step 3** Evaluate the new integral, if possible.
- Step 4** Express the antiderivative found in step 3 in terms of the original variable.

## 6. Integration

### 6-2 Integration by Substitution

#### EXAMPLE 4

Find  $\int (x^3 - 5)^4 3x^2 dx$

**Step 1** Select  $u$ .

Let  $u = x^3 - 5$ , then  $du = 3x^2 dx$

**Step 2** Express integral in terms of  $u$ .

$$\int (x^3 - 5)^4 3x^2 dx = \int u^4 du$$

**Step 3** Integrate.

$$\int u^4 du = \frac{u^5}{5} + C$$

**Step 4** Express the answer in terms of  $x$ .

$$\int (x^3 - 5)^4 3x^2 dx = \frac{(x^3 - 5)^5}{5} + C$$

## 6. Integration

### 6-2 Integration by Substitution

#### EXAMPLE 5

Find  $\int (x^2 + 5)^{\frac{1}{2}} 2x dx$

**Step 1** Select  $u$ .

Let  $u = x^2 + 5$ , then  $du = 2x dx$

**Step 2** Express integral in terms of  $u$ .

$$\int (x^2 + 5)^{\frac{1}{2}} 2x dx = \int u^{\frac{1}{2}} du$$

**Step 3** Integrate.

$$\int u^{\frac{1}{2}} du = \frac{2}{3} u^{\frac{3}{2}} + C$$

**Step 4** Express the answer in terms of  $x$ .

$$\int (x^2 + 5)^{\frac{1}{2}} 2x dx = \frac{2}{3} (x^2 + 5)^{\frac{3}{2}} + C$$

## 6. Integration

### 6-2 Integration by Substitution

#### EXAMPLE 6

Find  $\int (x^3 - 5)^4 x^2 dx$

**Step 1** Select  $u$ .

Let  $u = x^3 - 5$ , then  $du = 3x^2 dx$ , or  $dx = \frac{1}{3x^2} du$

**Step 2** Express integral in terms of  $u$ .

$$\int (x^3 - 5)^4 x^2 dx = \int u^4 \frac{x^2}{3x^2} du = \frac{1}{3} \int u^4 du$$

**Step 3** Integrate.

$$\frac{1}{3} \int u^4 du = \frac{1}{3} \frac{u^5}{5} + C$$

**Step 4** Express the answer in terms of  $x$ .

$$\int (x^3 - 5)^4 x^2 dx = \frac{1}{15} (x^3 - 5)^5 + C$$

## 6. Integration

### 6-2 Integration by Substitution

#### EXAMPLE 7

Find  $\int x^2 e^{4x^3} dx$

**Step 1** Select  $u$ .

Let  $u = 4x^3$ , then  $du = 12x^2 dx$ , or  $dx = \frac{1}{12x^2} du$

**Step 2** Express integral in terms of  $u$ .

$$\int x^2 e^{4x^3} dx = \int e^u \frac{x^2}{12x^2} du = \frac{1}{12} \int e^u du$$

**Step 3** Integrate.

$$\frac{1}{12} \int e^u du = \frac{1}{12} e^u + C$$

**Step 4** Express the answer in terms of  $x$ .

$$\int x^2 e^{4x^3} dx = \frac{1}{12} e^{4x^3} + C$$

## 6. Integration

### 6-2 Integration by Substitution

#### EXAMPLE 8

Find  $\int \frac{x}{(5-2x^2)^5} dx$

**Step 1** Select  $u$ .

Let  $u = 5 - 2x^2$ , then  $du = -4x dx$ , or  $dx = -\frac{1}{4x} du$

**Step 2** Express integral in terms of  $u$ .

$$\int \frac{x}{(5-2x^2)^5} dx = \int \frac{x}{u^5} \frac{-1}{4x} du = -\frac{1}{4} \int \frac{1}{u^5} du$$

**Step 3** Integrate.

$$-\frac{1}{4} \int u^{-5} du = (-\frac{1}{4}) \left( \frac{-1}{4} \right) u^{-4} + C$$

**Step 4** Express the answer in terms of  $x$ .

$$\int \frac{x}{(5-2x^2)^5} dx = \frac{1}{16(5-2x^2)^4} + C$$

## 6. Integration

6-2 Integration by Substitution

### EXAMPLE 9

Find  $\int x(x+6)^8 dx$

**Step 1** Let  $u = x + 6$ , then  $du = dx$ .

**Step 2** Express integral in terms of  $u$ .

$$\int x(x+6)^8 dx = \int \textcolor{red}{x} u^8 du$$

We need to get rid of the  $\textcolor{red}{x}$ , and express it in term of  $u$ :  $x = u - 6$ .

$$\int \textcolor{red}{x} u^8 du = \int (u - 6) u^8 du = \int (u^9 - 6u^8) du$$

**Step 3**

$$\int (u^9 - 6u^8) du = \frac{u^{10}}{10} - 6 \frac{u^9}{9} + C$$

**Step 4**

$$\int x(x+6)^8 dx = \frac{(x+6)^{10}}{10} - 2 \frac{(x+6)^9}{3} + C$$

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## 6. Integration

6-2 Integration by Substitution

### Application

The marginal price of a supply level of  $x$  bottles of baby shampoo per week is given by

$$p'(x) = \frac{300}{(3x+25)^2}$$

Find the price-supply equation if the distributor of the shampoo is willing to supply 75 bottles a week at a price of \$1.60 per bottle.

We need to find  $p(x)$

$$p(x) = \int p'(x) dx = \int \frac{300}{(3x+25)^2} dx$$

Let  $u = 3x + 25$ , then  $du = 3dx$  or  $dx = \frac{1}{3} du$ .

$$\begin{aligned} p(x) &= \int p'(x) dx = \int \frac{300}{(3x+25)^2} dx \\ &= \int \frac{300}{u^2} \frac{1}{3} du = 100 \int \frac{1}{u^2} du = \frac{-100}{u} + C \\ &= \frac{-100}{3x+25} + C \end{aligned}$$

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## 6. Integration

6-2 Integration by Substitution

### Application

Now we need to find  $C$  using the fact that 75 bottles sell for \$1.60 per bottle.

$$p(x) = \frac{-100}{3x+25} + C$$

$$p(75) = 1.60 = \frac{-100}{3(75)+25} + C = \frac{-100}{250} + C = -0.4 + C$$

$$C = 2$$

$$\text{So } p(x) = \frac{-100}{3x+25} + 2$$

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## 6. Integration

6-2 Integration by Substitution

### EXERCISES

1. Find each indefinite integral and check the result by differentiating.

$$a. \int 6x^5(x^6+1)^3 dx \quad b. \int \frac{x^2}{(5-x^3)^4} dx \quad c. \int 10x(x^2-1)^5 dx$$

$$d. \int 7e^{7x} dx \quad e. \int (x-1)e^{x^2-2x} dx \quad f. \int \frac{x}{x^2+2} dx$$

$$g. \int \frac{1}{(3t+4)^2} dt \quad h. \int x\sqrt{x+1} dx \quad i. \int \frac{x}{\sqrt{x-7}} dx$$

2. The weekly marginal revenue from the sale of  $x$  designer leather belts is given by

$$R'(x) = 30 - 0.001x + \frac{300}{x}, \quad R(1) = 50,$$

where  $R(x)$  is revenue in dollars.

Find the revenue function. Find the revenue from the sale of 500 leather belts.

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## 6. Integration

6-3 Differential Equations; Growth and Decay

### Learning Objectives

- Solve basic differential equations.
- Solve applications involving differential equations.

## 6. Integration

6-3 Differential Equations; Growth and Decay

### Intro to Differential Equations

We previously studied equations like

$$\begin{aligned}\frac{dy}{dx} &= 3x^2 - 5x + 3 \\ p'(x) &= 300e^{-0.05x} \\ y'' &= 3x^2 - 5x + 4\end{aligned}$$

These are examples of **differential equations**.

The first two are **first order differential equations** involving only the first derivative while the last is a **second order differential equation** because it involves the second derivative.

## 6. Integration

6-3 Differential Equations; Growth and Decay

### Differential Equations and Slope Fields

A **slope field** is a graphical representation of the solutions of a differential equation. It is useful because it can be created without solving the differential equation analytically. The representation may be used to qualitatively visualize solutions, or to numerically approximate them.

Slope fields will be introduced through an example.

Let

$$\frac{dy}{dx} = y + 1$$

Remember the geometric interpretation of the derivative is the slope of the line at the given point.

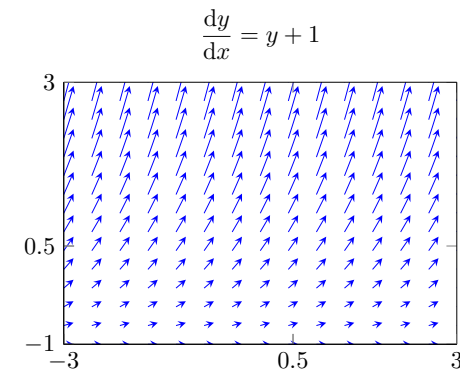
For the given function, the slope at the point  $(x, y) = (0, 2)$  would be 3. At the points  $(-2, 1)$  and  $(2, 3)$ , the slopes would be 2 and 4 respectively.

A **slope field** places a short line segment at each point on the graph indicating the slope of the function at that point.

## 6. Integration

6-3 Differential Equations; Growth and Decay

### Differential Equations and Slope Fields



The graph of the slope field for all points.

## 6. Integration

6-3 Differential Equations; Growth and Decay

### Differential Equations and Slope Fields

We claim that the solution of

$$\frac{dy}{dx} = y + 1$$

is

$$y = Ce^x - 1.$$

Let's check to see if it works. Substitute in the original equation for  $y$ .

$$\begin{aligned}\frac{d}{dx}(Ce^x - 1) &= Ce^x \\ &= (Ce^x - 1) + 1 \\ &= y + 1\end{aligned}$$

We have confirmed that  $y = Ce^x - 1$  is a solution to the differential equation  $\frac{dy}{dx} = y + 1$ .

## 6. Integration

6-3 Differential Equations; Growth and Decay

### Slope Fields in General

A **slope field** for a first-order differential equation is obtained by drawing tangent line segments determined by the equation at each point in a grid.

In general, this is done by computers, but to obtain a few points by hand we can do the following:

- Draw tangent lines for a solution curve of the differential equation that passes through a few points.
- Sketch an approximate graph of the solution curve that passes through these points.
- Of all the elementary functions previously discussed, make a conjecture as to what type of function appears to be a solution to the differential equation.

## 6. Integration

6-3 Differential Equations; Growth and Decay

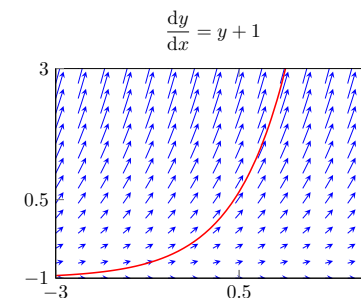
### Differential Equations and Slope Fields

$y = Ce^x - 1$  is the **general solution** to the differential equation  $\frac{dy}{dx} = y + 1$ .

A **particular solution** going through the point  $(0, 0)$  would be  $0 = Ce^0 - 1$  or  $C = 1$ . This yields the equation

$$y = e^x - 1$$

We will now graph that equation on our slope field:



## 6. Integration

6-3 Differential Equations; Growth and Decay

### Continuous Compound Interest Revisited

Let  $P$  be the initial amount of money deposited in an account ( $A(0) = P$ ).

Let  $A$  be the amount at any time  $t$ .

Continuous compound interest means that the rate of growth of the money in the account at any time  $t$  is proportional to the amount present at that time.

Since  $\frac{dA}{dt}$  is the rate of growth of  $A$  with respect to  $t$ , we have

$$\frac{dA}{dt} = rA$$

where  $r$  is an appropriate constant.

To find the function  $A = A(t)$  that satisfies the above conditions we divide both sides of the above equation by  $A$  and integrate with respect to time  $t$ .

$$\frac{1}{A} \frac{dA}{dt} = r$$

## 6. Integration

6-3 Differential Equations; Growth and Decay

### Continuous Compound Interest Revisited

Now we integrate each side with respect to  $t$ :

$$\begin{aligned}\int \frac{1}{A} \frac{dA}{dt} dt &= \int r dt \\ \int \frac{1}{A} dA &= \int r dt \\ \ln |A| &= rt + C \\ \ln A &= rt + C \\ A &= e^{rt+C} = e^C e^{rt}\end{aligned}$$

Since  $A(0) = P$ , we evaluate  $A(t) = e^C e^{rt}$  at  $t = 0$  and set the result equal to  $P$ :

$$A(0) = e^C e^0 = e^C = P$$

Hence,  $e^C = P$  and we can rewrite  $A(t) = e^C e^{rt}$  in the form  $A = P e^{rt}$ . This is the same continuous compound interest formula obtained in Section 5-1, where the principal  $P$  is invested at an annual nominal rate  $r$  compounded continuously for  $t$  years.

## 6. Integration

6-3 Differential Equations; Growth and Decay

### Relative Growth Rate

The constant  $r$  in the exponential growth law is called the **relative growth rate**.

If the relative growth rate is  $r = 0.02$ , then the quantity  $Q$  is growing at a rate

$\frac{dQ}{dt} = 0.02Q$  (that is 2% of the quantity  $Q$  per unit of time  $t$ ).

Note the distinction between the relative growth rate  $r$  and the rate of growth  $\frac{dQ}{dt}$  of the quantity  $Q$ .

Relative growth rate is  $0.02$  and the rate of growth is  $0.02Q$ .

Once we know that the rate of growth of something is proportional to the amount present, we know that it has exponential growth and we can use the exponential growth formula.

## 6. Integration

6-3 Differential Equations; Growth and Decay

### Exponential Growth Law

In general, the rate of growth of money in the previous case may be extended to any quantity  $Q$  that grows proportionally to the amount present with respect to time. We obtain the following theorem:

#### THEOREM 1 Exponential Growth Law

If  $\frac{dQ}{dt} = rQ$  and  $Q(0) = Q_0$ , then  $Q = Q_0 e^{rt}$ , where

- $Q_0$  = amount of money at  $t = 0$
- $r$  = relative growth rate (expressed as a decimal)
- $t$  = time
- $Q$  = quantity at time  $t$

If  $r > 0$ , then  $\frac{dQ}{dt} > 0$  and  $Q$  is increasing, this becomes **exponential growth**.

If  $r < 0$ , then  $\frac{dQ}{dt} < 0$  and  $Q$  is decreasing this becomes an **exponential decay** problem.

## 6. Integration

6-3 Differential Equations; Growth and Decay

### EXAMPLE 1

China had a population of 1.32 billion in 2007 ( $t = 0$ ). Let  $P$  represent the population (in billions)  $t$  years after 2007, and assume a continuous growth rate of 0.6%.

Find the estimated population for China in the year 2025.

The exponential growth/decay law applies, so that

$$P = 1.32e^{0.006t}$$

Substituting  $18 = 25 - 7$  for  $t$  yields

$$P = 1.32e^{0.006 \cdot 18} = 1.47$$

billion people

## 6. Integration

6-3 Differential Equations; Growth and Decay

### EXAMPLE 2

A bone from an ancient tomb was discovered and was found to have 5% of the original radioactive carbon present. The continuous compound rate of decay for radioactive carbon-14 is 0.0001238.

Estimate the age of the bone.

The exponential growth/decay law that applies is

$$\frac{dQ}{dt} = -0.0001238Q, \quad Q(0) = Q_0$$

The solution is

$$Q(t) = Q_0 e^{-0.0001238t} = 0.05Q_0$$

So

$$\begin{aligned} 0.05 &= e^{-0.0001238t} \\ \ln 0.05 &= \ln(e^{-0.0001238t}) \\ t &= \frac{\ln 0.05}{-0.0001238} \approx 24,198 \text{ years} \end{aligned}$$

## 6. Integration

6-4 The Definite Integral

### Learning Objectives

- Approximate areas using left and right sums.
- Use properties of definite integrals to solve problems.

## 6. Integration

6-3 Differential Equations; Growth and Decay

### EXERCISES

1. Find the general or particular solution, as indicated, for each differential equation.

$$a. \frac{dy}{dx} = e^{0.035x} \quad b. \frac{dy}{dx} = \sqrt[3]{x} \quad \frac{dy}{dx} = \frac{1}{3x-5}; y(2) = 3$$

2. Find the general or particular solution, as indicated, for each differential equation.

$$a. \frac{dy}{dx} = 4y \quad b. \frac{dx}{dt} = 0.3x; x(0) = 50$$

3. Show that  $y = Ce^{-x} + 3$  is a solution to the differential equation  $\frac{dy}{dx} = 3 - y$  for any real number  $C$ . Find the particular solution that passes through the point  $(0, 0)$ .

4. Find the amount  $A$  in an account after  $t$  years if  $\frac{dA}{dt} = 0.065A$  and  $A(0) = 10,000$ .

5. The marginal price  $\frac{dp}{dx}$  at  $x$  units of supply per day is proportional to the price  $p$ . There is no weekly demand at a price of \$300 per unit ( $p(0) = 300$ ), and there is a weekly demand of 10 units at a price \$250 per unit ( $p(10) = 250$ ).

- Find the price-demand equation.
- At a demand of 20 units per week, what is the price?

## 6. Integration

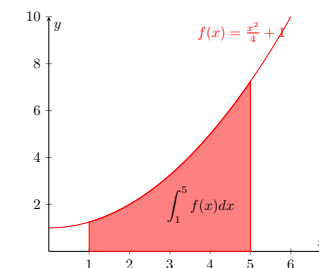
6-4 The Definite Integral

### Introduction

We have been studying the **indefinite integral** or **antiderivative** of a function. We will be interested in the **area** between graph of a function  $f(x)$  and the  $x$ -axis from  $x = a$  and  $x = b$ .

We now introduce the **definite integral**. This integral will be the area bounded by  $f(x)$ , the  $x$  axis, and the vertical lines  $x = a$  and  $x = b$ , with notation

$$\int_a^b f(x) dx$$



## 6. Integration

### 6-4 The Definite Integral

One way to approximate the area under a curve is by filling the region with rectangles and calculating the sum of the areas of the rectangles.

#### DEFINITION of left Sum

Given a function  $f(x)$  and an interval  $a \leq x \leq b$ ,  $[a, b]$  and a positive integer  $n$ , define

**Symbol:**  $L_n$

**Spoken:** the left sum with  $n$  rectangles.

#### Meaning

- Subdivide the interval  $[a, b]$  into  $n$  equal subintervals.
- On each subinterval, put a "left rectangle". That is, a rectangle that
  - Sits on the  $x$ -axis
  - Goes up or down until its left edge touches the graph of  $f$
- Add up the signed area of all the rectangles. The resulting sum is the value of  $L_n$ .

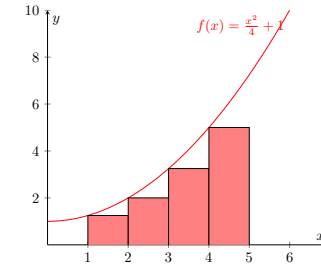
Analogous definition for the "Right sum",  $R_n$ , involving **Right** rectangles

## 6. Integration

### 6-4 The Definite Integral

#### Estimating Area

Take the width of each rectangle to be  $\Delta x = 1$  ( $\Delta x = \frac{b-a}{n} = \frac{5-1}{4} = 1$ ). Summing the areas of the left rectangles results in a left sum of four rectangles, denoted by  $L_4$ . If we use the left endpoints, the heights of the four rectangles are  $f(1), f(2), f(3), f(4)$ .



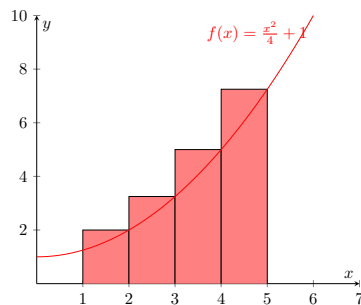
$$\begin{aligned} L_4 &= f(1)\Delta x + f(2)\Delta x + f(3)\Delta x + f(4)\Delta x \\ &= 1.25 + 2 + 3.25 + 5 \\ &= 11.5 \end{aligned}$$

## 6. Integration

### 6-4 The Definite Integral

#### Estimating Area

We can repeat this using the right side of each rectangle to determine the height. The heights of each of the four rectangles are now  $f(2), f(3), f(4)$  and  $f(5)$ , respectively.



$$\begin{aligned} R_4 &= f(2)\Delta x + f(3)\Delta x + f(4)\Delta x + f(5)\Delta x \\ &= 2 + 3.25 + 5 + 7.25 \\ &= 17.5 \end{aligned}$$

## 6. Integration

### 6-4 The Definite Integral

#### Estimating Area

**Observe:** If  $f(x)$  is increasing on the interval  $[a, b]$ , then the actual value of the (signed) Area will be sandwiched in between  $L_n$  and  $R_n$

$$L_n < \text{Area} = \int_a^b f(x)dx < R_n$$

Similarly, if  $f(x)$  is decreasing on  $[a, b]$ , then

$$R_n < \text{Area} = \int_a^b f(x)dx < L_n$$

In our example, we have

$$L_4 = 11.5 < \text{Area} = \int_1^5 f(x)dx < R_4 = 17.5$$

Note that the average of  $L_4$  and  $R_4$  would be an even better approximation:  
 $\text{Area} \approx (11.5 + 17.5)/2 = 14.5$

## 6. Integration

### 6-4 The Definite Integral

#### Estimating Area

The previous average of 14.5 is very close to the actual area of 14.333.... Our accuracy can be improved if we increase the number of rectangles, and let  $\Delta x$  get smaller.

The difference between the actual value of the definite integral and either the left or right Riemann sum is the **error** in that approximation.

But in both of the cases above, this error can be no larger than the (absolute value of the) difference between the left and right sums, since one is an underestimate and the other is an overestimate. Thus we have:

$$\text{Error}_{L_n} = |\text{Area} - L_n| \leq |R_n - L_n|$$

$$\text{Error}_{R_n} = |\text{Area} - R_n| \leq |R_n - L_n|$$

## 6. Integration

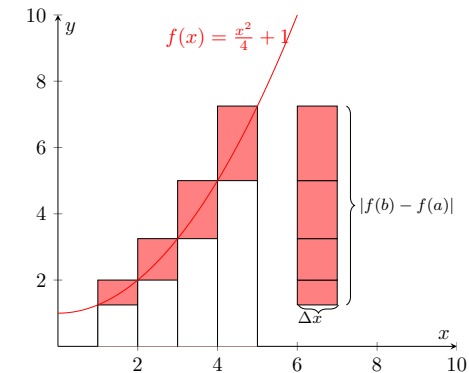
### 6-4 The Definite Integral

It is not hard to show that

$$|R_n - L_n| = |f(b) - f(a)|\Delta x$$

and that for  $n$  equal subintervals,  $\Delta x = \frac{b-a}{n}$ .

Difference between  $R_4$  and  $L_4$



## 6. Integration

### 6-4 The Definite Integral

#### THEOREM 1 Error Bounds for Approximations of Area by Left or Right Sums

If  $f(x) > 0$  and is either increasing on  $[a, b]$  or decreasing on  $[a, b]$ , then

$$|f(b) - f(a)| \frac{b-a}{n}$$

is an error bound for the approximation of the area, by  $L_n$  or  $R_n$ .

For our previous example:

$$\text{Error} \leq |R_4 - L_4| = |f(5) - f(1)| \cdot 1 = 6$$

#### THEOREM2 Limits of Left and Right Sums

If  $f(x) > 0$  and is either increasing on  $[a, b]$  or decreasing on  $[a, b]$ , then its left and right sums approach the same real number  $I$  as  $n \rightarrow \infty$

This number  $I$  is the area between the graph of  $f$  and the  $x$  axis from  $x = a$  to  $x = b$ .

## 6. Integration

### 6-4 The Definite Integral

#### Approximating Area

In our example, we had

$$\text{Error} \leq |7.25 - 1.25| \frac{5-1}{4} = 6$$

If we wanted a particular accuracy, say 0.05, we could use the error formula to calculate  $n$ , the number of rectangles needed:

$$|7.25 - 1.25| \frac{5-1}{n} = 0.05$$

Solving for  $n$  yields  $n = 480$ . We would need at least 480 rectangles to guarantee an accuracy of 0.05.

## 6. Integration

### 6-4 The Definite Integral

#### Definite Integral as Limit of Sums

We now come to a general definition of the definite integral.

Let  $f$  be a function on interval  $[a, b]$ . Partition  $[a, b]$  into  $n$  subintervals of equal length  $\Delta x = \frac{b-a}{n}$  with endpoints

$$a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$$

In each subinterval, choose an arbitrary point  $x_{k-1} < c_k < x_k$

Then, using summation notation we have

$$L_n = f(x_0)\Delta x + f(x_1)\Delta x + \dots + f(x_{n-1})\Delta x = \sum_{k=1}^n f(x_{k-1})\Delta x$$

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_n)\Delta x = \sum_{k=1}^n f(x_k)\Delta x$$

$$S_n = f(c_1)\Delta x + f(c_2)\Delta x + \dots + f(c_n)\Delta x = \sum_{k=1}^n f(c_k)\Delta x$$

$S_n$  is called a **Riemann sum**. Notice that  $L_n$  and  $R_n$  are both special cases of a Riemann sum.

## 6. Integration

### 6-4 The Definite Integral

#### THEOREM 3 Limit of Riemann Sums

If  $f$  is a continuous function on  $[a, b]$ , then the Riemann sums for  $f$  on  $[a, b]$  approach a real number limit  $I$  as  $n \rightarrow \infty$ .

This limit  $I$  of the Riemann sums for  $f$  on  $[a, b]$  is called the **definite integral** of  $f$  from  $a$  to  $b$ , denoted

$$\int_a^b f(x)dx$$

The **integrand** is  $f(x)$ , the **lower limit** of integration is  $a$ , and the **upper limit** of integration is  $b$ .

## 6. Integration

### 6-4 The Definite Integral

#### Area (Revisited)

Let's revisit our original problem and calculate the Riemann sum using the midpoints for  $c_k$  ( $c_k = \frac{x_{k-1} + x_k}{2}$ ).

The width of each rectangle is again  $\Delta x = 1$ . The heights are now  $f(\frac{3}{2}), f(\frac{5}{2}), f(\frac{7}{2}), f(\frac{9}{2})$ .

The sum of the rectangles is then

$$S_4 = 1.5625 + 2.5625 + 4.0625 + 6.0625 = 14.25$$

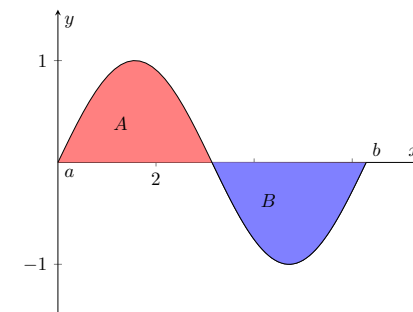
This is quite close to the actual area of 14.333...

## 6. Integration

### 6-4 The Definite Integral

#### Negative Values

If  $f(x)$  is positive for some values of  $x$  on  $[a, b]$  and negative for others, then the definite integral symbol  $\int_a^b f(x)dx$  represents the cumulative sum of the signed areas between the graph of  $f(x)$  and the  $x$ -axis, where areas above are positive and areas below negative.



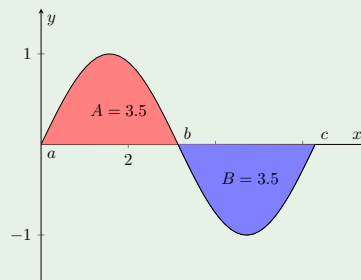
$$\int_a^b f(x)dx = A - B$$

## 6. Integration

6-4 The Definite Integral

### EXAMPLE 1

Calculate the definite integrals by referring to the figure with the indicated areas.



$$\int_a^a f(x)dx = 3.5$$

$$\int_b^c f(x)dx = -3.5$$

$$\int_a^c f(x)dx = 3.5 - 3.5 = 0$$

## 6. Integration

6-4 The Definite Integral

### PROPERTIES Properties of Definite Integrals

$$1. \int_a^a f(x)dx = 0$$

$$2. \int_a^b f(x)dx = - \int_b^a f(x)dx$$

$$3. \int_a^b k f(x)dx = k \int_a^b f(x)dx$$

$$4. \int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

$$5. \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

## 6. Integration

6-4 The Definite Integral

### EXERCISES

1. Calculate the indicated Riemann sum for the function  $f(x) = x^2 + 5x$ .

- Right sum  $R_3$  on  $[0, 3]$ .
- Left sum  $L_3$  on  $[0, 3]$ .
- Partion  $[0, 3]$  into 3 subintervals of equal length and let  $c_k = \frac{x_{k-1} + x_k}{2}$

2. Compute the error bounds for  $R_3$  and  $L_3$  if  $f(x) = x^2 + 5x$  on  $[0, 3]$

3. Calculate the following definite integrals given that:

$$\int_0^6 x dx = 18, \quad \int_0^6 x^2 dx = 72 \quad \int_3^6 x^2 dx = 63.$$

$$a. \int_0^6 3x dx \quad b. \int_0^6 (x - x^2) dx \quad c. \int_0^3 x^2 dx \quad \int_6^3 x^2 dx$$

4. How large must  $n$  be chosen so that  $\int_0^3 (x^2 + 5x)dx = R_n \pm 0.1$ ?

## 6. Integration

6-5 The Fundamental Theorem of Calculus

### Learning Objectives

- Use the Fundamental Theorem of Calculus to evaluate definite integrals.
- Find the average value of a function.



## 6. Integration

6-5 The Fundamental Theorem of Calculus

### THEOREM 1 Fundamental Theorem of Calculus

If  $f$  is a **continuous function** on the closed interval  $[a, b]$ , and  $F$  is any antiderivative of  $f$ , then

$$\int_a^b f(x)dx = F(b) - F(a)$$

### Evaluating Definite Integrals

By the fundamental theorem we can evaluate

$$\int_a^b f(x)dx$$

easily and exactly. We simply calculate

$$F(b) - F(a)$$

## 6. Integration

6-5 The Fundamental Theorem of Calculus

### PROPERTIES Properties of Definite Integrals

1.  $\int_a^a f(x)dx = 0$
2.  $\int_a^b f(x)dx = -\int_b^a f(x)dx$
3.  $\int_a^b kf(x)dx = k \int_a^b f(x)dx$
4.  $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
5.  $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

## 6. Integration

6-5 The Fundamental Theorem of Calculus

### EXAMPLE 1

$$\int_1^3 5dx = 5 \int_1^3 dx = 5x \Big|_1^3 = 5[3 - 1] = 5 \cdot 2 = 10$$

### EXAMPLE 2

$$\int_1^3 xdx = \frac{x^2}{2} \Big|_1^3 = \frac{3^2}{2} - \frac{1^2}{2} = 4$$

### EXAMPLE 3

$$\int_0^3 x^2 dx = \frac{x^3}{3} \Big|_0^3 = \frac{3^3}{3} - \frac{0^3}{3} = 9 - 0 = 9$$

## 6. Integration

6-5 The Fundamental Theorem of Calculus

### EXAMPLE 4 (Definite Integrals and Substitution)

$$\int_{-1}^1 e^{2x} dx = \frac{1}{2} \int_{-2}^2 e^u du = \frac{e^u}{2} \Big|_{-2}^2 = \frac{e^2}{2} - \frac{e^{-2}}{2}$$

If  $u = 2x$ , then  $du = 2dx$ , and

$x = -1$  implies  $u = -2$

$x = 1$  implies  $u = 2$

### EXAMPLE 5

$$\int_1^2 \frac{1}{x} dx = \ln x \Big|_1^2 = \ln 2 - \ln 1 = \ln 2$$

## 6. Integration

6-5 The Fundamental Theorem of Calculus

### EXAMPLE 6 (Definite Integrals and Substitution)

$$\int_0^5 \frac{x^2}{x^3 + 4} dx = \frac{1}{3} \int_4^{129} \frac{1}{u} du = \frac{\ln u}{3} \Big|_4^{129} = \frac{\ln 129}{3} - \frac{\ln 4}{3}$$

Let  $u = x^3 + 4$ , then  $du = 3x^2 dx$ , and

$x = 0$  implies  $u = 4$

$x = 5$  implies  $u = 129$

## 6. Integration

6-5 The Fundamental Theorem of Calculus

### EXAMPLE 7

From past records a management service determined that the rate of increase in maintenance cost for an apartment building (in dollars per year) is given by

$$M'(x) = 90x^2 + 5,000,$$

where  $M(x)$  is the total accumulated cost of maintenance for  $x$  years.

Write a definite integral that will give the total maintenance cost from the end of the second year to the end of the seventh year. Evaluate the integral.

$$\int_3^7 (90x^2 + 5,000) dx = 30x^3 + 5000x \Big|_3^7 = 12,290 + 35000 - 810 - 15,000 = \$29,480$$

## 6. Integration

6-5 The Fundamental Theorem of Calculus

### Using Definite Integrals for Average Values

The **average value** of a continuous function  $f$  over  $[a, b]$  is

$$\frac{1}{b-a} \int_a^b f(x) dx$$

Note this is the area under the curve divided by the width.  
Hence, the result is the average height or average value.

## 6. Integration

6-5 The Fundamental Theorem of Calculus

### EXAMPLE 8

The total cost (in dollars) of printing  $x$  dictionaries is

$$C(x) = 20,000 + 10x$$

- Find the average cost per unit if 1000 dictionaries are produced.
- Find the average value of the cost function over the interval  $[0, 1000]$ .
- Write a description of the difference between part a. and part b.

## 6. Integration

6-5 The Fundamental Theorem of Calculus

### EXAMPLE 8

a. The average cost is

$$\bar{C}(x) = \frac{C(x)}{x} = \frac{20000}{x} + 10$$

$$\bar{C}(1000) = \frac{20000}{1000} + 10 = 30$$

b. Find the average value of the cost function over the interval  $[0, 1000]$ .

$$\begin{aligned} \frac{1}{b-a} \int_a^b C(x) dx &= \frac{1}{1000} \int_0^{1000} (20,000 + 10x) dx \\ &= \frac{1}{1000} (20,000x + 5x^2) \Big|_0^{1000} \\ &= \frac{1}{1000} (20,000 \cdot 1000 + 5(1000)^2) \\ &= 20,000 + 5,000 = 25,000 \end{aligned}$$

c. Write a description of the difference between part a. and part b.

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## 6. Integration

6-5 The Fundamental Theorem of Calculus

### EXAMPLE 8

c. If you just do the set-up for printing, it costs \$20,000. This is the cost for printing 0 dictionaries.

If you print 1,000 dictionaries, it costs \$30,000. That is \$30 per dictionary (part a).

If you print some random number of dictionaries (between 0 and 1000), on average it costs \$25,000 (part b).

Those two numbers really have not much to do with one another.

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## 6. Integration

6-5 The Fundamental Theorem of Calculus

### EXERCISES

1. Evaluate the following definite integrals.

$$a. \int_0^4 x dx, \quad b. \int_{-1}^1 (1 - x^3) dx, \quad c. \int_0^1 e^{7x} dx, \quad d. \int_1^5 \frac{x+2}{x^2+4x} dx$$

2. Find the average value of :

$$a. g(x) = x^2 + 4 \text{ on } [-2, 2] \quad b. h(x) = \sqrt{x} \text{ on } [0, 9]$$

3. The total cost (in dollars) of making  $x$  music boxes is given by  $C(x) = 12,000 + 40x$

a. Find the average cost per unit if 200 music boxes are produced.

b. Find the average value of the cost function on the interval  $[0, 200]$ .

c. Explain the difference in the meaning of the values found in parts a and b.

4. A company produces a printer that also scans documents. The research department produced the marginal cost function  $C'(x) = 200 - \frac{x}{5}$  where  $C(x)$  is the total cost (in dollars) and  $x$  is the number of printers produced in a month. Compute the increase in cost going from a production level of 100 printers per month to 500 printers per month. Set up a definite integral and evaluate.

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## 7. Additional Integration Topics

7-1 Area Between Curves

### Learning Objectives

- Determine the area of a region bound by two or more curves.
- Solve applications pertaining to income distribution.

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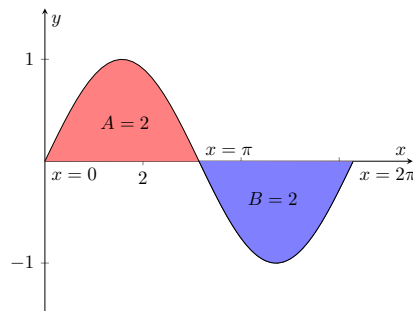
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## 7. Additional Integration Topics

### 7-1 Area Between Curves

#### Negative Values for the Definite Integral

Remember the idea of **unsigned area** and **signed area**.



- **Unsigned Area** between graph of  $f(x)$  and the  $x$ -axis between  $x = 0$  and  $x = 2\pi$  is  $USA = A + B = 2 + 2 = 4$ .
- **Signed Area** is  $SA = A - B = 2 - 2 = 0 = \int_{x=0}^{x=2\pi} f(x)dx$  = the definite integral.

In this sec we'll study the "area between curves" these words mean the **unsigned area**.

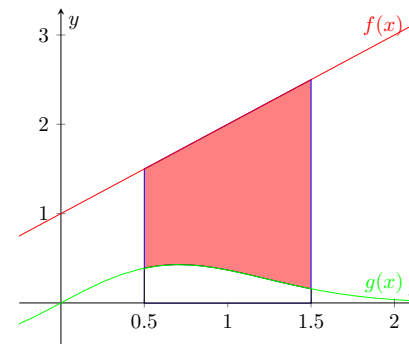
## 7. Additional Integration Topics

### 7-1 Area Between Curves

#### THEOREM 1 Area Between Two Curves

If  $f(x) \geq g(x)$  on some interval  $[a, b]$ , then (unsigned) area bounded by  $y = f(x)$  and  $y = g(x)$  for  $a \leq x \leq b$  (in other words the area between the graph of  $f$  and  $g$  from  $x = a$  to  $x = b$ ) is given by

$$\int_{x=a}^{x=b} [f(x) - g(x)] dx$$



## 7. Additional Integration Topics

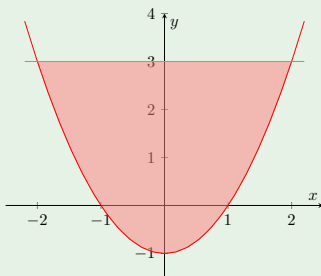
### 7-1 Area Between Curves

#### EXAMPLE 1

Find the area bounded by

$$y = x^2 - 1 \text{ and } y = 3.$$

Note the two curves intersect at  $x = \pm 2$ , and  $y = 3$  is the larger function on  $[-2, 2]$ .



$$\int_{-2}^2 [3 - (x^2 - 1)] dx = \int_{-2}^2 [4 - x^2] dx = 4x - \frac{x^3}{3} \Big|_{-2}^2 = 4(2) - \frac{2^3}{3} - 4(-2) + \frac{-2^3}{3} = \frac{32}{3}$$

## 7. Additional Integration Topics

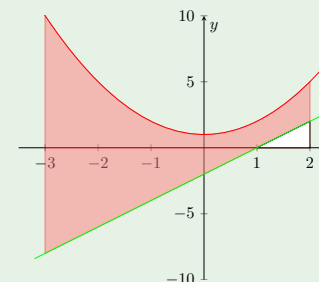
### 7-1 Area Between Curves

#### EXAMPLE 2

Find the area between the curves

$$y = x^2 + 1 \text{ and } y = 2x - 2.$$

from  $x = -3$  to  $x = 2$ .



$$\int_{-3}^2 [(x^2 + 1) - (2x - 2)] dx = \int_{-3}^2 (x^2 - 2x + 3) dx = \left( \frac{x^3}{3} - x + 3x \right) \Big|_{-3}^2 = \frac{95}{3}$$

## 7. Additional Integration Topics

### 7-1 Area Between Curves

#### Application: Income Distribution

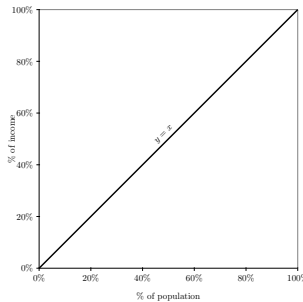
The quantity of interest is the yearly income per family.

One hypothetical extreme: every family has the **same** yearly income. "**absolute equality**".

Consider making a graph

$x$ -axis: the percentage of the families in the country that are included.

$y$ -axis: the percentage of the country's total yearly income that those families earn.



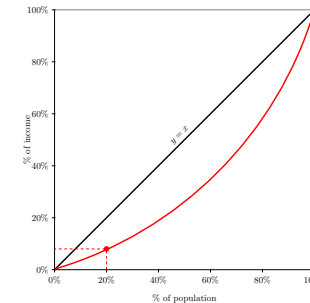
## 7. Additional Integration Topics

### 7-1 Area Between Curves

#### Application: Income Distribution

Now suppose a different scenario: some families make more money per year than others.

$x$ -axis: the % of the families in the country that are included, with the lowest income family included **first**.



Data point (20, 8) indicates that the bottom 20% of families receive only 8% of the total income

## 7. Additional Integration Topics

### 7-1 Area Between Curves

#### Gini Index

A measure of the inequality of income distribution is

$$\frac{\text{area between the curves}}{\text{the area the red curve}} = \frac{\text{area between the curves}}{\frac{1}{2}bh} = \text{The Gini Index}$$

The diagram shows two small graphs. The first graph shows the area between the line y=x and a red curve, shaded in purple. The second graph shows the area under the red curve, shaded in red. The red curve is a parabola y=x^2.

$$\text{area between the curves} = \int_{x=0}^{x=1} (x - f(x))dx = \int_{x=0}^{x=1} xdx - \int_{x=0}^{x=1} f(x)dx$$

The diagram shows two small graphs. The first graph shows the area under the line y=x from x=0 to x=1, shaded in red. The second graph shows the area under the red curve y=x^2 from x=0 to x=1, shaded in green.

A measure of 0 indicates absolute equality. A measure of 1 indicates absolute inequality (one family has all the income, the rest have none).

## 7. Additional Integration Topics

### 7-1 Area Between Curves

#### Gini Index

But the area of the red triangle is

$$A = \frac{1}{2}b \cdot h = \frac{1}{2}(1)(1) = \frac{1}{2}$$

A small graph showing the area under the line y=x from x=0 to x=1, shaded in red. The area is a triangle with base 1 and height 1.

So the area between the curves is

$$\frac{1}{2} - \int_{x=0}^{x=1} f(x)dx = \frac{1}{2} - \int_{x=0}^{x=1} x^2 dx = \frac{1}{2} - \frac{1}{3}$$

The diagram shows three small graphs. The first graph shows the area under the line y=x from x=0 to x=1, shaded in red. The second graph shows the area under the red curve y=x^2 from x=0 to x=1, shaded in green. The third graph shows the area between the line y=x and the red curve y=x^2, shaded in purple.

So the Gini Index will be:

$$G = \frac{\text{area between the curves}}{\text{the area the red curve}} = \frac{\frac{1}{2} - \int_{x=0}^{x=1} f(x)dx}{\frac{1}{2}} = 2 \left( \frac{1}{2} - \int_{x=0}^{x=1} f(x)dx \right)$$

$$= 1 - 2 \int_{x=0}^{x=1} f(x)dx$$

## 7. Additional Integration Topics

### 7-1 Area Between Curves

#### EXAMPLE 1

Suppose the curve of income distribution is given by  $f(x) = x^8$ . Find the Gini Index and draw a picture.

$$\begin{aligned} G &= 1 - 2 \int_{x=0}^{x=1} f(x) dx = 1 - 2 \int_{x=0}^{x=1} x^8 dx = 1 - 2 \left( \frac{x^9}{9} \right) \Big|_{x=0}^{x=1} \\ &= 1 - 2 \left( \frac{1^9}{9} - \frac{0^9}{9} \right) = 1 - \frac{2}{9} = \frac{7}{9} \end{aligned}$$

## 7. Additional Integration Topics

### 7-1 Area Between Curves

#### EXAMPLE 2

A country is planning changes in tax structure in order to provide a more equitable distribution of income. The two Lorenz curves are:

$$\begin{aligned} f(x) &= x^{2.3} \text{ currently, and} \\ g(x) &= 0.4x + 0.6x^2 \text{ proposed.} \end{aligned}$$

Will the proposed changes work?

Currently: Gini Index of income concentration=

$$1 - 2 \int_{x=0}^{x=1} x^{2.3} dx = 1 - 2 \left( \frac{x^{3.3}}{3.3} \right) \Big|_{x=0}^{x=1} = 1 - 2 \left( \frac{1^{3.3}}{3.3} - \frac{0^{3.3}}{3.3} \right) = 1 - \frac{2}{3.3} \approx 0.3939$$

Future: Gini Index of income concentration=

$$1 - 2 \int_0^1 0.4x + 0.6x^2 dx = 1 - 2 \left( 0.4 \frac{x^2}{2} + 0.6 \frac{x^3}{3} \right) \Big|_0^1 = 1 - 2(0.2x^2 + 0.2x^3) \Big|_0^1 = 0.20$$

The Gini index is decreasing, so the future distribution will be more equitable.

## 7. Additional Integration Topics

### 7-1 Area Between Curves

#### EXERCISES

1. Find the area bounded by the graphs of the indicated equations over the given interval. Compute answers to three decimal places when necessary.

- $y = 2x - 1, y = 5, -2 \leq x \leq 2$
- $y = 3 - x^2, y = x - 2, -1 \leq x \leq 1$
- $y = e^{2x}, y = 2 - x, 1 \leq x \leq 3$

2. Find the area bounded by the graphs of the indicated equations over the given intervals (when stated). Compute answers to three decimal places when necessary.

- $y = -x^2 + 6, y = x$
- $y = x^3 - 3x^2, y = x^2$
- $y = x^4 - 3x^2, y = 7x^2 - 9$

3. Suppose that the following Lorenz curves represent the distribution of income for a small country in 2005 and 2008, respectively:  $f(x) = x^2 + 0.4x - 0.5$ ,  $g(x) = 0.25x^2 + 0.6x + 0.01$ . Find the Gini index of the income concentration for both Lorenz curves and interpret the results.